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JEE MASTER BASH
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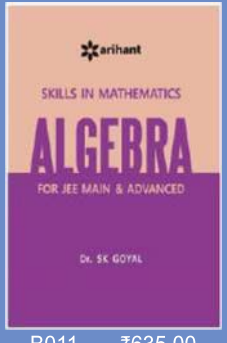
FOR JEE MAIN & ADVANCED

BY DR. SK GOYAL & AMIT M AGARWAL



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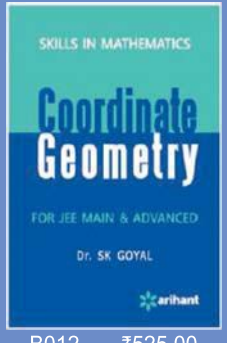

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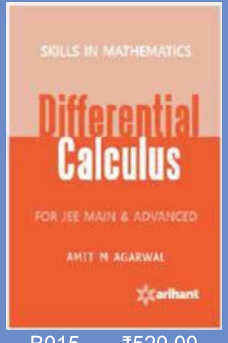

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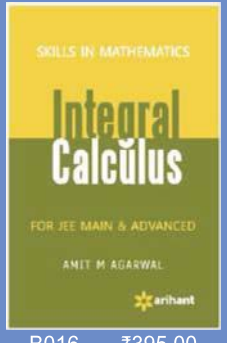

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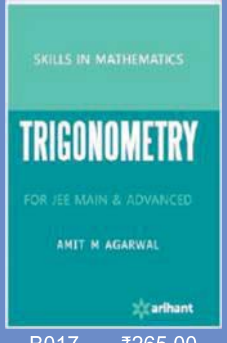

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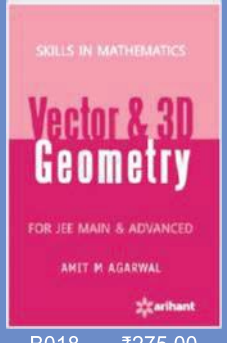

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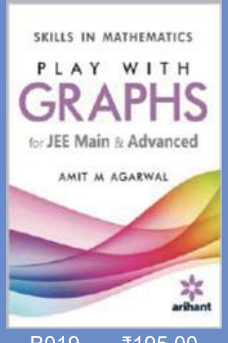

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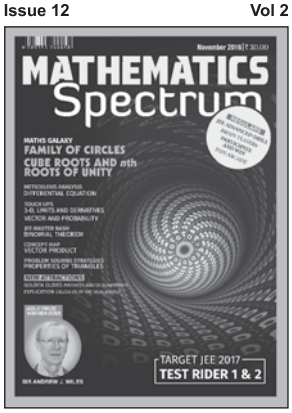
Issue 12 **Vol 2**



Dear Aspirants,

By now your studies would be in full swing as the season of examinations is approaching. Hope you had a gala festival time and a good break to enjoy. The month of November begins with different vibes altogether. Pinkish winters, new energies, overflowing enthusiasm and increased confidence levels give altogether a different turn to life.

Do you know, the word algebra is derived from the Arabic word al-jabar. This word is a part of the title of a 9th century work "hisab-al-jabar w'al-muquablah" written by "Mohammad ibn Musa al-khowarizmi". The word al-jabar in itself signifies "restoration", i.e. a reference to the fact that, if a number is added to one side of an equation, then it must be added to the other side as well to restore the equality. Likewise, calculus is a master piece of the human mind. It is a vast body of knowledge and an armory of analytical tools. It occupies a special place in our world because this is very helpful in describing everything – right from orbiting satellite and falling price to decaying atoms and spreading disease. So, Mathematics has an inherent difference with all the other branches of Science. This can be understood *via*. following precious lines of Vinzenz Earth

"God is a child; and when he began to play, he cultivated Mathematics. It is the most Godly of man's games."

After the month of festivals, it is the beginning of study time for all of you. We have also planned something special for you like Test Rider to make your task achievable. We will enhance the same from next issue onwards. The spectrum team will be anxiously waiting for the feed backs and comments.

Good luck

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The Abel Prize (Norwegian: Abelprisen) is a Norwegian prize awarded annually by the Government of Norway to one or more outstanding mathematicians. It is named after Norwegian mathematician Niels Henrik Abel (1802–29) and modelled after the Nobel Prizes. The award was established in 2001 by the Government of Norway and complements its sister prize in the humanities, i.e. the Holberg Prize. It comes with a monetary award of 6 million Norwegian kroner (NOK) (around € 600,000).

WINNER OF ABLE PRIZE OF MATHEMATICS 2016

This years Able Prize was awarded to Sir Andrew J. Wiles **for his stunning proof of Fermat's Last Theorem by way of the modularity conjecture for semi stable elliptic curves. This will open a new era in number theory.**

FERMAT LAST THEOREM

This theorem was first conjectured by Pierre de Fermat in 1637 in the *Margin of a copy of Arithmetica* where he claimed he had a proof that was too large to fit in the margin. The first successful proof was released in 1994 by Andrew Wiles, and formally published in 1995, after 358 years of effort by mathematicians. It is among the most notable theorems in the history of mathematics and prior to its proof, it was in the Guinness Book of World Records as the "most difficult mathematical problem", one of the reasons being that it has the largest number of unsuccessful proofs. **Fermat's Last Theorem** (sometimes called Fermat's conjecture, especially in older texts) states that no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two. The cases $n = 1$ and $n = 2$ have been known to have infinitely many solutions since antiquity. In simpler words the theorem is that there are no solutions, in positive integers a, b, c to the equation $a^n + b^n = c^n$, once $n > 2$ (there are lots of solutions when $n = 2$).

THE ABEL PRIZE LAUREATES 2003-2015

2015 : John F. Nash Jr. & Louis Nirenberg
2014 : Yakov G. Sinai
2013 : Pierre Deligne
2012 : Endre Szemerédi
2011 : John Milnor
2010 : John Torrence Tate
2009 : Mikhail Leonidovich Gromov
2008 : John Griggs Thompson and Jacques Tits
2007 : Srinivasa S. R. Varadhan
2006 : Lennart Carleson
2005 : Peter D. Lax
2004 : Sir Michael Francis Atiyah and Isadore M. Singer
2003 : Jean-Pierre Serre

WILES CONTRIBUTION

Andrew Wiles's proof of Fermat's Last Theorem is a proof of the modularity theorem for semi stable which, together with Ribet's theorem, provides a proof for Fermat's Last Theorem. Wiles actually proved a special case of the Shimura-Tanayama-Weil modularity conjecture, which *via* some very difficult theorems implies Fermat's Last Theorem. The proof itself is over 150 pages long and consumed seven years of Wiles's research time. John Coates described the proof as one of the highest achievements of number theory, and John Conway called it the proof of the century. Wiles's techniques led to further developments in number theory, such as the proof of the general case of the modularity conjecture in 2001, and the impact of his work is still being felt today.

NIELS HENRIK ABEL

The exceptional Norwegian mathematician Niels Henrik Abel was born on 5 August 1802. When he died, only 26 years old, he left a large body of work. He is most famous for proving the impossibility of solving the general quintic equation via radicals. You probably remember the quadratic formula for finding solutions of polynomials of degree 2, and there are similar formulas for polynomials of degrees 3 and 4; Abel showed there is no such formula for polynomials of degree 5 or higher. He had a significant impact on mathematics beyond that also in his short 26-year life. Many fundamental concepts bear his name like abelian groups, abelian functions, abelian varieties along with the first proof of the general binomial theorem, which had been stated by Newton and Euler.

Love Agarwal

FAMILY OF CIRCLES

In this article, we will learn about family of circles, common tangents and angle of intersection of two circles. These are very important concept which are frequently asked in JEE.

Family of Circles

A collection of circles is called a system or family of circles. There can be various types of family of circles. The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Since, this equation involves three unknowns, *viz.*, g, f and c, so we need atleast three conditions to get a unique circle.

Theorem 1

The equation of a family of circles passing through the intersection of a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and line

$$L \equiv lx + my + n = 0 \text{ is}$$

$$x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$$

or

$$S + \lambda L = 0$$

where, λ is any real number.

Example 1 The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is

(a) $x^2 + y^2 + x + y = 0$ (b) $x^2 + y^2 - x - y = 0$
(c) $x^2 + y^2 - x + y = 0$ (d) $x^2 + y^2 + x - y = 0$

Sol. ***(b)*** Equation of any circle passing through the point of intersection of $x^2 + y^2 - 2x = 0$ and $y = x$ is

$$S + \lambda L = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x + \lambda(y - x) = 0$$

$$\Rightarrow \quad x^2 + y^2 - (2 + \lambda)x + \lambda y = 0$$

Now, above circle has centre $\left[\left(\frac{2+\lambda}{2}\right), \frac{-\lambda}{2}\right]$.

For AB to be the diameter of the required circle, the centre must be on AB.

$$\therefore \quad \frac{2+\lambda}{2} = \frac{-\lambda}{2} \quad \Rightarrow \quad \lambda = -1$$

Therefore, equation of the required circle is

$$x^2 + y^2 - (2 - 1)x - y = 0$$

$$\Rightarrow \quad x^2 + y^2 - x - y = 0$$

Example 2 Equation of the smallest circle passing through the intersection of the line $x + y = 1$ and the circle $x^2 + y^2 = 9$ is

(a) $x^2 + y^2 - x - y - 8 = 0$
(b) $x^2 + y^2 + x + y + 9 = 0$
(c) $x^2 + y^2 + 2x + 2y + 4 = 0$
(d) $2x^2 + 2y^2 - 2x - 2y - 7 = 0$

Sol. **Key point** Smallest circle is that circle having given line as diameter.

(a) Any circle passing through the point of intersection of the given line and circle has the equation

$$x^2 + y^2 - 9 + \lambda(x + y - 1) = 0 \quad \ldots(i)$$

Its centre is $\left(\frac{-\lambda}{2}, \frac{-\lambda}{2}\right)$.

This circle is the smallest, if $\left(\frac{-\lambda}{2}, \frac{-\lambda}{2}\right)$ is on the chord $x + y - 1 = 0$.

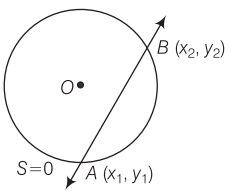
$$\therefore \quad \frac{-\lambda}{2} - \frac{\lambda}{2} - 1 = 0 \Rightarrow \lambda = -1$$

On putting the value of λ in Eq. (i), the equation of the smallest circle is

$$x^2 + y^2 - 9 - 1(x + y - 1) = 0$$
$$\Rightarrow \quad x^2 + y^2 - x - y - 8 = 0$$

Theorem 2

The equation of the family of circles passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is



$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

or $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda L = 0$, where $L = 0$ represents the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ and $\lambda \in R$.

POINTS TO REMEMBER

1. If $x_2 \to x_1$ and $y_2 \to y_1$, then equation of the circle becomes $(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$.
2. Also, when $x_2 \to x_1$ and $y_2 \to y_1$, then line AB becomes tangent to the circle $S = 0$.

Example 3 The circle passing through $(1, -2)$ and touching the axis of X at $(3, 0)$, also passes through the point **[JEE Main 2013]**

(a) $(-5, 2)$ (b) $(2, -5)$ (c) $(5, -2)$ (d) $(-2, 5)$

Sol. **(c)** The equation of the circle touching X-axis, i.e. $y = 0$ at $(3, 0)$ is

$$(x - 3)^2 + (y - 0)^2 + \lambda y = 0 \quad \ldots(i)$$

$\because$ It passes through $(1, -2)$.

$$\therefore \quad (1 - 3)^2 + (-2 - 0)^2 - 2\lambda = 0$$
$$\Rightarrow \quad \lambda = 4$$

Putting $\lambda = 4$ in Eq. (i), we obtain

$$(x - 3)^2 + y^2 + 4y = 0$$
$$\Rightarrow \quad x^2 + y^2 - 6x + 4y + 9 = 0$$

Clearly, it passes through $(5, -2)$.

Example 4 If C_1, C_2 and C_3 belong to a family of circles through the points (x_1, y_1) and (x_2, y_2), then the ratio of lengths of the tangent from any point on C_1 to the circles is

(a) constant (b) cannot be determined
(c) equal to difference of x-coordinates
(d) equal to difference of y-coordinates

Sol. **(a)** Equation of the family of circles through (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Let (h, k) be the point on C_1.

$$\Rightarrow \quad f(h, k) + \lambda_1 g(h, k) = 0$$

where, $f(h, k) = (h - x_1)(h - x_2) + (k - y_1)(k - y_2)$

and $$g(h, k) = \begin{vmatrix} h & k & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

Let T_2 be the length of the tangent from (h, k) to C_2 and T_3 be the length of the tangent from (h, k) to C_3.

Then, $T_2 = \sqrt{f(h, k) + \lambda_2 g(h, k)}$

and $T_3 = \sqrt{f(h, k) + \lambda_3 g(h, k)}$

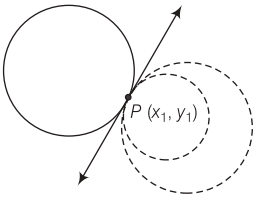
$$\therefore \quad \frac{T_2}{T_3} = \frac{\sqrt{f(h, k) + \lambda_2 g(h, k)}}{\sqrt{f(h, k) + \lambda_3 g(h, k)}}$$

$$\Rightarrow \quad \frac{T_2}{T_3} = \frac{\sqrt{(\lambda_2 - \lambda_1) g(h, k)}}{\sqrt{(\lambda_3 - \lambda_1) g(h, k)}}$$

$$\Rightarrow \quad \frac{T_2}{T_3} = \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}} = \text{Constant}$$

Theorem 3

The equation of the family of circles touching the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, y_1)$ is



$$x^2 + y^2 + 2gx + 2fy + c + \lambda [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c] = 0$$

or $S + \lambda L = 0$, where $L = 0$ is the equation of the tangent to $S = 0$ at (x_1, y_1) and $\lambda \in R$.

POINTS TO REMEMBER

1. The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + k\,[y - y_1 - m(x - x_1)] = 0$, where k is a parameter.
2. In case the line through (x_1, y_1) is parallel to Y-axis, the equation of the family of circles touching it at (x_1, y_1) becomes $(x - x_1)^2 + (y - y_1)^2 + k(x - x_1) = 0$.
3. Also, if the line parallel to X-axis, then the equation of family of circles touching it at $(x_1, y_1) = 0$ becomes $(x - x_1)^2 + (y - y_1)^2 + k(y - y_1) = 0$.

Example 5 The circle passing through the point $(-1, 0)$ and touching the Y-axis at $(0, 2)$, also passes through the point [IIT JEE 2011]

(a) $\left(-\frac{3}{2}, 0\right)$ (b) $\left(-\frac{5}{2}, 2\right)$ (c) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (d) $(-4, 0)$

Sol. ***(d)*** The equation of the family of circles touching Y-axis at $(0, 2)$ is

$$(x - 0)^2 + (y - 2)^2 + \lambda x = 0$$

It passes through $(-1, 0)$.

$$1 + 4 - \lambda = 0 \Rightarrow \lambda = 5$$

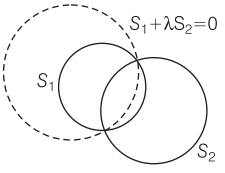
So, the equation of the circle is

$$x^2 + 5x + y^2 - 4y + 4 = 0$$

Clearly, it passes through $(-4, 0)$.

Theorem 4

The equation of a family of circles passing through the intersection of two circles



$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is $S_1 + \lambda S_2 = 0$

i.e. $$(x^2 + y^2 + 2g_1x + 2f_1y + c_1) + \lambda\,(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0,$$

where $\lambda\,(\neq -1)$ is an arbitrary real number.

Example 6 The equation of the circle passing through $(1, -3)$ and the points common to the two circles $x^2 + y^2 - 6x + 8y - 16 = 0$, $x^2 + y^2 + 4x - 2y - 8 = 0$ is

(a) $x^2 + y^2 - 4x + 6y + 24 = 0$
(b) $2x^2 + 2y^2 + 3x + y - 20 = 0$
(c) $3x^2 + 3y^2 - 5x + 7y - 19 = 0$
(d) None of the above

Sol. ***(b)*** The equation of circle is

$$(x^2 + y^2 - 6x + 8y - 16) + \lambda(x^2 + y^2 + 4x - 2y - 8) = 0$$

$\because$ It passes through $(1, -3)$.

$$\therefore \quad (1 + 9 - 6 - 24 - 16) + \lambda(1 + 9 + 4 + 6 - 8) = 0$$

$$\Rightarrow \quad -36 + \lambda(12) = 0$$

$$\Rightarrow \quad \lambda = 3$$

So, the equation of the required circle is

$$4x^2 + 4y^2 + 6x + 2y - 40 = 0$$

$$\Rightarrow \quad 2x^2 + 2y^2 + 3x + y - 20 = 0$$

Circle in Contact

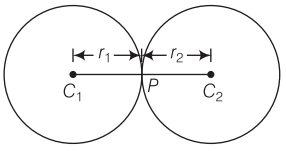
Let us discuss condition for two circles touching each other. Two circles can touch each other externally or internally.

Case I When Circles Touch Externally

When two circles touch each other externally, then distance between their centres is equal to sum of their radii,

i.e. $|C_1C_2| = r_1 + r_2$

In this case, point of contact P divides the line joining C_1 and C_2 internally in the ratio $r_1 : r_2$.



$$\Rightarrow \quad \frac{C_1P}{C_2P} = \frac{r_1}{r_2}$$

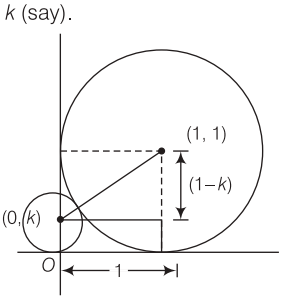
If $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$, then the coordinates of P is

$$\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right).$$

Example 7 Let C be the circle with centre $(1, 1)$ and radius 1 unit. If T is the circle centred at $(0, y)$ passing through origin and touching the circle C externally, then the radius of T is equal to [IIT JEE 2014]

(a) $\frac{\sqrt{3}}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Sol. ***(d)*** Here, the radius of circle C is 1 unit and the radius of another circle T whose centre is $(0, y)$ and passing through the origin is $y = k$ (say).



From the figure,

Distance between the centres = Sum of radii

$$\sqrt{1+(1-k)^2} = k+1$$

$$\Rightarrow \quad 1+1+k^2-2k = k^2+1+2k$$

$$\Rightarrow \quad 4k = 1 \Rightarrow k = \frac{1}{4}$$

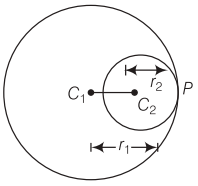
Hence, radius of circle T is $\frac{1}{4}$.

Case II When Circles Touch Internally

When two circles touch each other internally, then distance between their centres is equal to the difference of their radii,

i.e. $|C_1C_2| = |r_1 - r_2|$

In such a case, point of contact P divide the line C_1 and C_2 enternally in the ratio $r_1 : r_2$.



$$\Rightarrow \quad \frac{C_1P}{C_2P} = \frac{r_1}{r_2}$$

If $C_1 \equiv (x_1, y_1)$ and $C_2 \equiv (x_2, y_2)$, then the coordinates of P is $\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$.

Example 8 Two circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, then $\frac{1}{a^2} + \frac{1}{b^2}$ is equal to

(a) 1 (b) $\frac{1}{c^2}$

(c) 0 (d) None of these

Sol. **(b)** Given circles are

$$x^2 + y^2 + 2ax + c^2 = 0 \quad ...(i)$$

and $$x^2 + y^2 + 2by + c^2 = 0 \quad ...(ii)$$

Let C_1 and C_2 be the centres of circles (i) and (ii) respectively and r_1, r_2 be their radii, then $C_1(-a, 0)$, $C_2(0, -b)$, $r_1 = \sqrt{a^2 - c^2}$, $r_2 = \sqrt{b^2 - c^2}$. Since, the circles touch each other externally or internally.

$$\therefore \quad |C_1C_2| = |r_1 \pm r_2|$$

$$\Rightarrow \quad \sqrt{a^2 + b^2} = \left|\sqrt{a^2 - c^2} \pm \sqrt{b^2 - c^2}\right|$$

On squaring both the sides, we get

$$a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)(b^2 - c^2)}$$

$$\Rightarrow \quad c^2 = \pm\sqrt{(a^2 - c^2)(b^2 - c^2)}$$

$$\Rightarrow \quad c^2 = \pm\sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$$

$$\Rightarrow \quad c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$$

$$\Rightarrow \quad c^2(a^2 + b^2) = a^2b^2 \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$$

Common Tangents to Two Circles

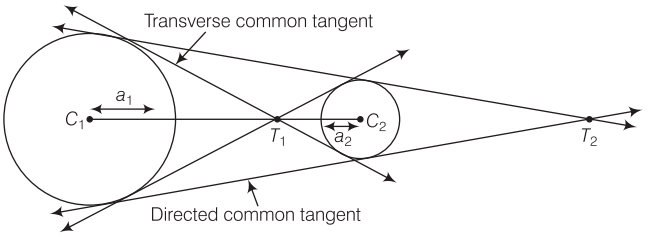
The number of common tangents depends upon the position of the given circles as discussed below. Let the equations of the two circles be

$$(x - h_1)^2 + (y - k_1)^2 = a_1^2 \quad ...(i)$$

and $$(x - h_2)^2 + (y - k_2)^2 = a_2^2 \quad ...(ii)$$

with centres $C_1(h_1, k_1)$ and $C_2(h_2, k_2)$, and radii a_1 and a_2, respectively. Then, the following cases of intersection of these two circles may arises.

Case I When $C_1C_2 > a_1 + a_2$

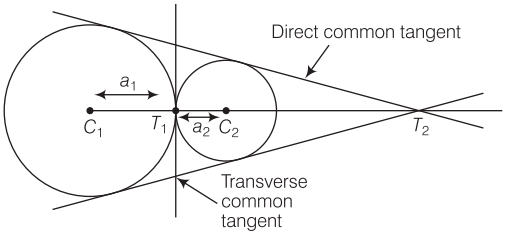


There are four common tangents that can be drawn to two circles. Two of them are direct common tangents and the other two are transverse common tangents. The points of intersection of direct and transverse common tangents always lie on the line joining the centres C_1 and C_2, and divide it internally and externally respectively in the ratio $a_1 : a_2$,

i.e. $\frac{C_1T_2}{C_2T_2} = \frac{a_1}{a_2}$ (externally) and $\frac{C_1T_1}{C_2T_1} = \frac{a_1}{a_2}$ (internally).

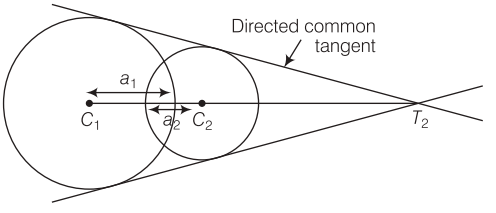
Case II When $C_1C_2 = a_1 + a_2$

In this case, two direct tangents are real and distinct while the transverse tangents are coincident.



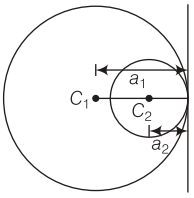
Case III When $C_1C_2 < a_1 + a_2$

In this case, the two direct common tangents are real while the transverse tangents are imaginary.



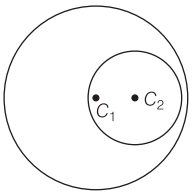
Case IV When $C_1C_2 = a_1 - a_2$

In this case, two tangents are real and coincident while the other two tangents are imaginary.



Case V When $C_1C_2 < a_1 - a_2$

In this case, all the four common tangents are imaginary.



Example 9 How many common tangents can be drawn to the circles $x^2 + y^2 = 6x$ and $x^2 + y^2 + 6x + 2y + 1 = 0$?

(a) 4 (b) 3
(c) 2 (d) 1

Sol. ***(a)*** The coordinates of the given circles are $C_1(3, 0)$, $C_2(-3, -1)$ and corresponding radii are $r_1 = 3$ and $r_2 = 3$, respectively.

Now, $C_1C_2 = \sqrt{(3+3)^2 + (0+1)^2} = \sqrt{37}$

and $r_1 + r_2 = 3 + 3 = 6$

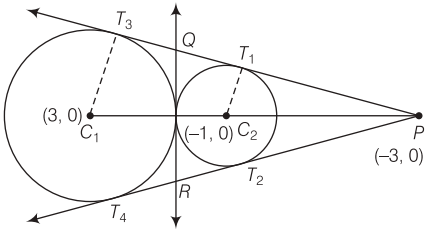
$\therefore$ $C_1C_2 > r_1 + r_2$

So, 4 common tangents can be drawn to the given circles.

Example 10 The common tangents to the circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 - 6x = 0$ form a triangle which is

(a) equilateral
(b) isosceles
(c) right angled
(d) None of the above

Sol. ***(b)*** Clearly, the distance between the centres of the given circles is equal to the sum of their radii. So, two given circles will touch each other externally.



We have,

$PT_1 = PT_2$ and $PT_3 = PT_4$

$\Rightarrow$ $T_1T_3 = T_2T_4$

$\Rightarrow$ $2T_1Q = 2T_2R$ $[\because QT_3 = QT_1$ and $RT_4 = RT_2]$

$\Rightarrow$ $T_1Q = T_2R$

$\Rightarrow$ $PT_1 + T_1Q = PT_1 + T_2R$

$\Rightarrow$ $PT_1 + T_1Q = PT_2 + T_2R$

$\Rightarrow$ $PQ = PR$

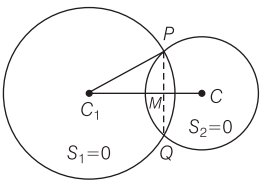
Hence, PQR is an isosceles triangle.

POINTS TO REMEMBER

1. Length of direct common tangent $= \sqrt{d^2 - (r_1 - r_2)^2}$.
2. Length of transverse common tangent $= \sqrt{d^2 - (r_1 + r_2)^2}$ where, d is distance between the circles and r_1, r_2 are radii.

Common Chord

The chord joining the points of intersection of two given circles is called their common chord. The equation of the common chord of two circles



$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is $$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$$

i.e. $$S_1 - S_2 = 0.$$

Length of the Common Chord

From above figure, we have

$$PQ = 2PM = 2\sqrt{C_1P^2 - C_1M^2}$$

where, C_1P = Radius of the circle $S = 0$

and C_1M = Length of the perpendicular from the centre C_1 to the common chord PQ

Example 11 If the circle $x^2 + y^2 + 6x - 2y + k = 0$ bisect the circumference of the circle $x^2 + y^2 + 2x - 6y - 15 = 0$, then k is equal to

(a) 21 (b) – 21
(c) 23 (d) – 23

Sol. ***(d)*** The equation of common chord of the given circle is

$$4x + 4y + (k + 15) = 0 \quad \ldots(i)$$

Since, $x^2 + y^2 + 6x - 2y + k = 0$ bisects the circumference of $x^2 + y^2 + 2x - 6y - 15 = 0$. Therefore, Eq. (i) is a diameter of the circle $x^2 + y^2 + 2x - 6y - 15 = 0$.

$\therefore$ $4(-1) + 4 \times 3 + k + 15 = 0$

$\Rightarrow$ $k = -23$

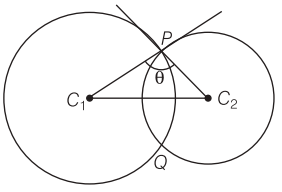
Angle of Intersection of Two Circles

The angle of intersection of two circles is defined as the angle between the tangents to the two circles at their point of intersection.

Let $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

be two circles having their centres at $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$.



Then, $\cos\theta = \dfrac{C_1P^2 + C_2P^2 - C_1C_2^2}{2C_1P \cdot C_2P}$

Orthogonal Circles

Two circles are said to intersect orthogonally, if their angles of intersection is a right angle.

For two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ cut orthogonally, the necessary condition is $2(g_1g_2 + f_1f_2) = C_1 + C_2$.

Example 12 The angle of intersection of the circles $x^2 + y^2 - 6x + 4y + 11 = 0$ and $x^2 + y^2 - 4x + 6y + 9 = 0$ is

(a) 30° (b) 45°
(c) 60° (d) 90°

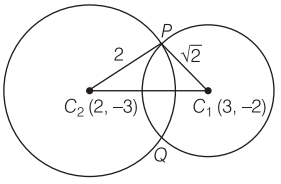
Sol. **(b)** The equations of the circles are

$x^2 + y^2 - 6x + 4y + 11 = 0$ ...(i)

and $x^2 + y^2 - 4x + 6y + 9 = 0$ ...(ii)

The centres of these circles are $C_1(3, -2)$ and $C_2(2, -3)$, respectively. Let r_1 and r_2 be their radii.

Then, $r_1 = \sqrt{9 + 4 - 11} = \sqrt{2}$ and $r_2 = \sqrt{4 + 9 - 9} = 2$



Also, $C_1C_2 = \sqrt{2}$

Suppose circles (i) and (ii) intersect at P.

Then,

$$\cos(\angle C_1PC_2) = \frac{C_1P^2 + C_2P^2 - C_1C_2^2}{2C_1P \cdot C_2P} = \frac{2 + 4 - 2}{2 \times \sqrt{2} \times 2} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \angle C_1PC_2 = 45°$

Smart Practice

1. C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0), respectively. C_3 is a circle of unit radius. It passes through the centres of the circles C_1 and C_2, and its centre above the X-axis. Then, the equation of the common tangent to C_1 and C_3 which does not pass through C_2, is

(a) $\sqrt{3}x + y - 2 = 0$ (b) $\sqrt{3}y + x - 2 = 0$
(c) $\sqrt{3}x - y + 2 = 0$ (d) $\sqrt{3}y - x - 2 = 0$

2. The radius of the smallest circle which touches the straight line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$, is (compute upto one place of decimal only)

(a) 2.5 (b) 1.5 (c) 3.5 (d) 4.5

3. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is

(a) $8\sqrt{3}$ sq units (b) $4\sqrt{3}$ sq units
(c) $6\sqrt{3}$ sq units (d) None of these

4. Two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents ($a > b \geq 2$). Then,

(a) $\theta = 2\cos^{-1}\left(\dfrac{a-b}{a+b}\right)$

(b) $\theta = 2\tan^{-1}\left(\dfrac{a+b}{a-b}\right)$

(c) $\theta = 2\sin^{-1}\left(\dfrac{a+b}{a-b}\right)$

(d) $\theta = 2\sin^{-1}\left(\dfrac{a-b}{a+b}\right)$

5. The angle at which the circles $(x-1)^2 + y^2 = 10$ and $x^2 + (y-2)^2 = 5$ intersect, is

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$
(c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

6. If the line $x\cos\theta + y\sin\theta = 2$ is the equation of a transverse common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6\sqrt{3}x - 6y + 20 = 0$, then the value of θ is

(a) $\dfrac{5\pi}{6}$ (b) $\dfrac{2\pi}{4}$
(c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{6}$

7. If $C_1 : x^2 + y^2 - 20x + 64 = 0$ and $C_2 : x^2 + y^2 + 30x + 144 = 0$, then the length of the shortest line segment PQ which touches C_1 at P and C_2 at Q, is

(a) 20 (b) 15
(c) 22 (d) 27

8. The coordinates of two points P and Q are (x_1, y_1) and (x_2, y_2) and O is the origin. If the circles are described on OP and OQ as diameters, then the length of their common chord is

(a) $\frac{|x_1y_2 + x_2y_1|}{PQ}$ (b) $\frac{|x_1y_2 - x_2y_1|}{PQ}$
(c) $\frac{|x_1x_2 + y_1y_2|}{PQ}$ (d) $\frac{|x_1x_2 - y_1y_2|}{PQ}$

9. The equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 4x - 2y = 8$ and $x^2 + y^2 - 2x - 4y = 8$ and the point $(-1, 4)$ is

(a) $x^2 + y^2 + 4x + 4y - 8 = 0$
(b) $x^2 + y^2 - 3x + 4y + 8 = 0$
(c) $x^2 + y^2 + x + y - 8 = 0$
(d) $x^2 + y^2 - 3x - 3y - 8 = 0$

10. If the length of the common chord of two circles $x^2 + y^2 + 8x + 1 = 0$ and $x^2 + y^2 + 2\mu y - 1 = 0$ is $2\sqrt{6}$, then the value of μ are

(a) ± 2 (b) ± 3
(c) ± 4 (d) None of these

11. A circle C_1 of radius b touches the circle $x^2 + y^2 = a^2$ externally and has its centre on the positive X-axis, another circle C_2 of radius c touches the circle C_1 externally and has its center on the positive X-axis. Given $a < b < c$, the three circles have a common tangent, if a, b and c are in

(a) AP (b) GP
(c) HP (d) None of these

12. Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. The chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Then, the coordinates of this point are

(a) $\left(\frac{7}{2}, \frac{9}{2}\right)$ (b) $\left(\frac{9}{2}, \frac{7}{2}\right)$
(c) $\left(-\frac{7}{2}, \frac{9}{2}\right)$ (d) $\left(\frac{9}{2}, -\frac{7}{2}\right)$

13. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is equal to

(a) 2 or $-\frac{3}{2}$ (b) -2 or $-\frac{3}{2}$
(c) 2 or $\frac{3}{2}$ (d) -2 or $\frac{3}{2}$

14. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is [JEE Main 2015]

(a) 3 (b) 4
(c) 1 (d) 2

15. If the circles $x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - d = 0$, then $c + d$ is equal to

(a) 60 (b) 50
(c) 40 (d) 56

16. The circles having radii r_1 and r_2 intersect orthogonally. The length of their common chord is

(a) $\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$ (b) $\frac{\sqrt{r_1^2 + r_2^2}}{2r_1r_2}$
(c) $\frac{r_1r_2}{\sqrt{r_1^2 + r_2^2}}$ (d) $\frac{\sqrt{r_1^2 + r_2^2}}{r_1r_2}$

17. The distance from the centre of the circle $x^2 + y^2 = 2x$ to the common chord of the circles $x^2 + y^2 + 5x - 8y + 1 = 0$ and $x^2 + y^2 - 3x + 7y - 25 = 0$ is

(a) 2 (b) 4
(c) $\frac{34}{13}$ (d) $\frac{26}{17}$

18. If the radii of the circle $(x-1)^2 + (y-2)^2 = 1$ and $(x-7)^2 + (y-10)^2 = 4$ are increasing uniformly w.r.t. time as 0.3 unit/s and 0.4 unit/s, then they will touch each other at t equal to

(a) 45 s (b) 90 s
(c) 11 s (d) 135 s

19. A circle is given by $x^2 + (y-1)^2 = 1$. Another circle C touches it externally and also the X-axis. Then, the locus of its centre is

(a) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y \le 0\}$
(b) $\{(x, y) : x^2 + (y-1)^2 = 4\} \cup \{(x, y) : y \le 0\}$
(c) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y \le 0\}$
(d) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$

20. A circle S passes through the point $(0, 1)$ and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then,

(a) radius of S is 8
(b) radius of S is 7
(c) centre of S is $(-7, 11)$
(d) centre of S is $(-8, 1)$

Do You Know

We all know 0 was invented by Aryabhatt. As far as the invention of digits 1-9 is concerned, these are believed to be invented in Arab. These digits are also known as Arabic Numerals. The first positional numerical system was developed in Babylon in the 2nd Millenium BC.

Answers with Explanation

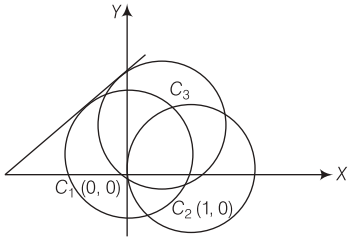
1. ***(c)*** The equation of any circle through (0, 0) and (1, 0) is

$$(x-1)(x-0)+(y-0)(y-0)+\lambda\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad x^2+y^2-x+\lambda y=0$$

If it represents C_3, then its radius is 1.

$$\therefore \quad 1=\frac{1}{4}+\frac{\lambda^2}{4} \quad \text{or} \quad \lambda=\pm\sqrt{3}$$



As the centre of C_3 lies above the X-axis. We take $\lambda=-\sqrt{3}$. Thus, an equation of C_3 is $x^2+y^2-x-\sqrt{3}y=0$. Since, C_1 and C_3 intersect and are of unit radius, their common tangents are parallel to the line joining their centres (0, 0) and $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

So, let the equation of a common tangent be $\sqrt{3}x-y+k=0$.

It will touch C_1, if $\frac{|k|}{\sqrt{3+1}}=1 \Rightarrow k=\pm 2$

2. ***(b)*** The equation of the circle touching the line $3x-y-6=0$ at $(1,-3)$ is given by

$$(x-1)^2+(y+3)^2+\lambda(3x-y-6)=0$$

$$\Rightarrow x^2+y^2+(3\lambda-2)x+(6-\lambda)y+10-6\lambda=0 \qquad \ldots(i)$$

$\because$ Eq. (i) touches $y=x$. Therefore,

$$2x^2+(2\lambda+4)x+10-6\lambda=0$$

i.e. $$x^2+(\lambda+2)x+5-3\lambda=0 \qquad \ldots(ii)$$

has equal roots. Therefore,

$$(\lambda+2)^2-4(5-3\lambda)=\lambda^2+16\lambda-16=0$$

$$\Rightarrow \quad \lambda=\frac{1}{2}[-16\pm\sqrt{256+64}]=-8\pm\sqrt{80}$$

Now, $(\text{Radius})^2=R^2$

$$=\frac{1}{4}[(3\lambda-2)^2+(6-\lambda)^2-(10-6\lambda)4]$$

$$=\frac{1}{4}[10\lambda^2]$$

$$\Rightarrow \quad R=\frac{\sqrt{10}\lambda}{2}=\frac{\sqrt{10}}{2}(-8\pm 4\sqrt{5})$$

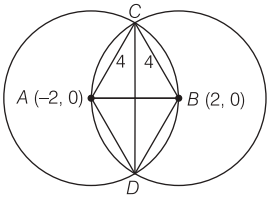
$$=\left|-4\sqrt{10}\pm 10\sqrt{2}\right|$$

$\therefore$ Radius of smaller circle $=\left|-4\sqrt{10}+10\sqrt{2}\right|$

$$=1.5 \text{ (approx.)}$$

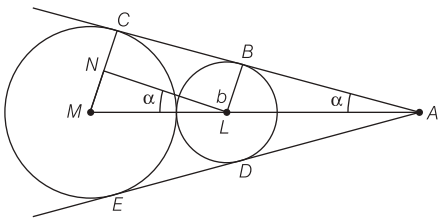
3. ***(a)*** The circles are with centres (2, 0) and (– 2, 0) and radius 4. Therefore, the Y-axis is their common chord.

$\therefore ABC$ is an equilateral triangle.



Hence, area of $ADBC=\frac{2\times\sqrt{3}}{4}(4)^2=8\sqrt{3}$ sq units

4. ***(d)***



From ΔMLN,

$$\sin\alpha=\frac{a-b}{a+b}$$

$$\Rightarrow \quad \alpha=\sin^{-1}\left(\frac{a-b}{a+b}\right)$$

$\therefore$ Angle between AB and $AD=2\alpha=2\sin^{-1}\left(\frac{a-b}{a+b}\right)$

5. ***(b)*** For given $r_1=\sqrt{10}$, $C_1\equiv(1,0)$

and for $\quad r_2=\sqrt{5}$, $C_2\equiv(0,2)$

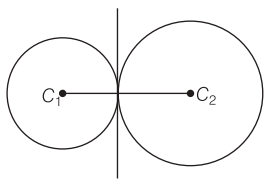
$\therefore \quad d=C_1C_2=\sqrt{5}$

If θ is the angle between the circles, then

$$\cos\theta=\frac{\left|d^2-r_1^2-r_2^2\right|}{2r_1r_2}=\frac{|5-10-5|}{2\sqrt{10}\sqrt{5}}=\frac{1}{\sqrt{2}}$$

Hence, $\quad \theta=\frac{\pi}{4}$

6. ***(d)*** $C_1C_2=r_1+r_2$



$C_1\equiv(0,0)$; $C_2\equiv(3\sqrt{3},3)$ and $r_1=2$, $r_2=4$

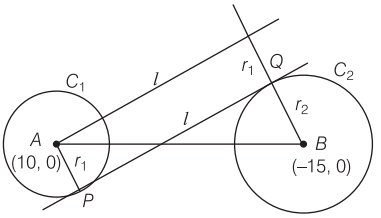
The circles touch each other externally.

The equation of common tangent is $\sqrt{3}x+y-4=0$.

Comparing it with $x\cos\theta+y\sin\theta=2$, we get $\theta=\frac{\pi}{6}$.

7. ***(a)*** The centres are (10, 0) and (– 15, 0), and the radii are $r_1 = 6$ and $r_2 = 9$.

Also, $d = 25, r_1 + r_2 < d$



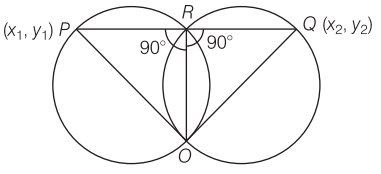
So, the circles are neither intersecting nor touching.

Therefore, $PQ = \sqrt{d^2 - (r_1 + r_2)^2}$

$= \sqrt{625 - 225} = \sqrt{400}$

$= 20$

8. ***(b)*** $OR = \dfrac{2 \times \text{Area of } \Delta OPQ}{PQ}$



$$= \frac{2 \times \left| \frac{1}{2}(x_1 y_2 - x_2 y_1) \right|}{PQ}$$

$$= \frac{|x_1 y_2 - x_2 y_1|}{PQ}$$

9. ***(d)*** The equation of any circle through the points of intersection of the given circles is

$x^2 + y^2 - 4x - 2y - 8$

$+ k(x^2 + y^2 - 2x - 4y - 8) = 0$ ...(i)

Since, circle (i) passes through (– 1, 4), we have $k = 1$

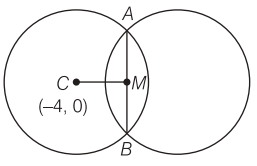
Therefore, the required circle is

$x^2 + y^2 - 3x - 3y - 8 = 0.$

10. ***(b)*** The common chord is $S_1 - S_2 = 0$.

i.e. $4x - \mu y + 1 = 0$

$\therefore$ AC = Radius of first circle $= \sqrt{16}$



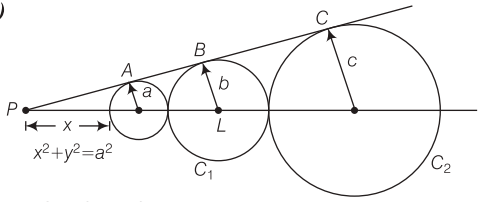
Also, $AM = \dfrac{AB}{2} = \sqrt{6}$

$\therefore$ $CM = 3$

$\Rightarrow$ $\dfrac{|-16 + 1|}{\sqrt{16 + \mu^2}} = 3$

$\Rightarrow$ $\mu = \pm 3$

11. ***(b)***



From the above figure,

$$\frac{a}{x} = \frac{b}{x + a + b} = \frac{c}{x + a + 2b + c}$$

$\therefore$ $ax + a^2 + ab = bx$ [comparing first two terms]

$\Rightarrow$ $x = \dfrac{a^2 + ab}{b - a}$

$\Rightarrow$ $ax + a^2 + 2ab + ac = cx$

[comparing first and third terms]

$\Rightarrow$ $x = \dfrac{a^2 + 2ab + ac}{c - a}$

$\therefore$ $\dfrac{a^2 + ab}{b - a} = \dfrac{a^2 + 2ab + ac}{c - a}$

$\Rightarrow a^2c - a^3 + abc - a^2b = a^2b + 2ab^2 + abc - a^3 - 2a^2 - b - a^2c$

$\Rightarrow$ $ac = b^2$

Therefore, a, b and c are in GP.

12. ***(a)*** The equation of the line passing through the points $A(3, 7)$ and $B(6, 5)$ is

$$y - 7 = -\frac{2}{3}(x - 3) \text{ or } 2x + 3y - 27 = 0$$

Also, the equation of the circle with A and B as the end points of diameter is

$(x - 3)(x - 6) + (y - 7)(y - 5) = 0$

Now, the equation of the family of circles through A and B is

$(x - 3)(x - 6) + (y - 7)(y - 5) + \lambda(2x + 3y - 27) = 0$ ...(i)

If the circle belonging to this family touches the X-axis.

Then, the equation

$(x - 3)(x - 6) + (0 - 7)(0 - 5) + \lambda\{2x + 3(0) - 27\} = 0$

has two equal roots for which discriminant $D = 0$.

It gives two values of λ.

The equation of the common chord of circle (i) and $x^2 + y^2 - 4x - 6y - 3 = 0$ is the radical axis which is

$[(x - 3)(x - 6) + (y - 7)(y - 5) + \lambda(2x + 3y - 27)] - [x^2 + y^2 - 4x - 6y - 3] = 0$

$\Rightarrow$ $(2\lambda - 5)x + (3\lambda - 6)y + (-27\lambda + 56) = 0$

$\Rightarrow$ $(-5x - 6y + 56) + \lambda(2x + 3y - 27) = 0$

This is the family of lines which passes through the point of intersection of $-5x - 6y + 56 = 0$ and $2x + 3y - 27 = 0$,

i.e. $\left(2, \dfrac{23}{3}\right)$.

If circle (i) cuts $x^2 + y^2 = 29$ orthogonally, then

$0 + 0 = -29 + 56 - 27\lambda = 0$ or $\lambda = 1$

So, the required circle is $x^2 + y^2 - 7x - 9y + 26 = 0$ and the centre is $\left(\dfrac{7}{2}, \dfrac{9}{2}\right)$.

13. ***(a)*** Given circles will intersect orthogonally, if

$$2(1 \times 0 + k \times k) = 6 + k$$

$$\Rightarrow \quad 2k^2 - k - 6 = 0$$

$$\Rightarrow \quad (2k + 3)(k - 2) = 0$$

$$\Rightarrow \quad k = 2, -\frac{3}{2}$$

14. ***(a)*** The coordinates of the centres and radii of the given circles are

$$C_1 : (2, 3) \quad C_2 : (-3, -9)$$

$$r_1 : 5 \quad r_2 : 8$$

We find $r_1 + r_2 = C_1C_2 = 13$. So, the two circles touch each other externally. Hence, there are 3 common tangents.

15. ***(b)*** The common chord of the given circles is

$$6x + 14y + c + d = 0 \qquad ...(i)$$

Since, $x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - d = 0$. So, the common chord (i) passes through the centre of the second circle, i.e. $(1, -4)$.

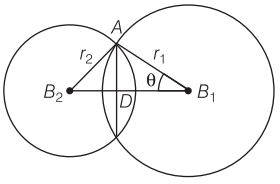
$$\therefore \quad 6 - 56 + c + d = 0 \Rightarrow c + d = 50$$

16. ***(a)*** Let $\angle AB_1B_2 = \theta$. Then,

$$AD = r_1 \sin\theta \text{ and } AD = r_2 \cos\theta$$

$$\therefore \quad AD^2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) = 1$$

$$\Rightarrow \quad AD = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$



Thus, length of common chord is $\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

17. ***(a)*** The centre of the circle $x^2 + y^2 = 2x$ is $(1, 0)$. The common chord of the other two circles is

$$8x - 15y + 26 = 0$$

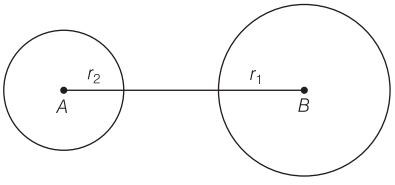
The distance from $(1, 0)$ to $8x - 15y + 26 = 0$ is

$$\frac{|8 + 26|}{\sqrt{15^2 + 8^2}} = 2$$

18. ***(b)*** Given circles are

$$(x - 1)^2 + (y - 2)^2 = 1 \qquad ...(i)$$

and $$(x - 7)^2 + (y - 10)^2 = 4 \qquad ...(ii)$$



Let $A \equiv (1, 2), B \equiv (7, 10), r_1 = 1, r_2 = 2,$

$$AB = 10, r_1 + r_2 = 3$$

$$\because \quad AB > r_1 + r_2$$

Hence, the two circles are separated.

The radii of the two circles at time t are $(1 + 0.3t)$ and $(2 + 0.4t)$.

For the two circles to touch each other,

$$AB^2 = [(r_1 + 0.3t) \pm (r_2 + 0.4t)]^2$$

$$\Rightarrow \quad 100 = [(1 + 0.3t) \pm (2 + 0.4t)]^2$$

$$\Rightarrow \quad 100 = (3 + 0.7t)^2, [(0.1)t + 1]^2$$

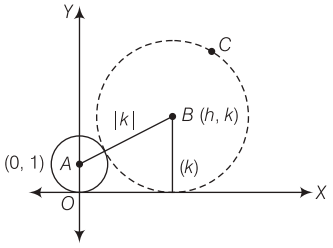
$$\Rightarrow \quad 3 + 0.7t = \pm 10, 0.1t + 1 = \pm 10$$

$$\therefore \quad t = 10, t = 90 \qquad [\because t > 0]$$

The two circles will touch each other externally in 10 s and internally in 90 s.

19. ***(d)*** Let the centre of the circle C be (h, k).

Then, as the circle touches the X-axis, its radius is $|k|$.



Also, it touches the given circle $x^2 + (y - 1)^2 = 1$ with centre $(0, 1)$ and radius 1, externally. Therefore,

Distance between centres = Sum of radii

$$\Rightarrow \quad \sqrt{(h - 0)^2 + (k - 1)^2} = 1 + |k|$$

$$\Rightarrow \quad h^2 + k^2 - 2k + 1 = 1 + 2|k| + k^2$$

$$\Rightarrow \quad h^2 = 2k + 2|k|$$

Therefore, the locus of (h, k) is $x^2 = 2y + 2|y|$.

Now, if $y > 0$, it becomes $x^2 = 4y$.

Also, if $y \le 0$, it becomes $x = 0$.

Combining the two, the required locus is

$$\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$$

20. ***(b)*** Let the circle S be $x^2 + y^2 + 2gx + 2fy + c = 0$. It is orthogonal to the circles $x^2 + y^2 - 2x - 15 = 0$ and $x^2 + y^2 - 1 = 0$.

Therefore, $2[g(-1) + f \times 0] = c - 15$

and $$2[g \times 0 + f \times 0] = c - 1$$

$$\Rightarrow \quad -2g = c - 15 \text{ and } c = 1$$

$$\Rightarrow \quad c = 1, g = 7$$

The circle S passes through $(0, 1)$.

$$\therefore \quad 1 + 2f + c = 0 \Rightarrow f = -1$$

Thus, the centre of the circle S is $(-7, 1)$ and radius is $\sqrt{49 + 1 - 1} = 7$.

'0' IS EITHER POSITIVE OR NEGATIVE

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Pravabati College,
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ABOUT THE CHARACTER OF ZERO

0 is the integer immediately preceding 1. Zero is an even number because it is divisible by 2. Also, 0 is neither positive nor negative. By most definitions, 0 is number which quantifies a count or an amount of null size. In most cultures 0 was identified before the idea of negative things (quantities) that go lower than zero was accepted.

The number 0 is the smallest non-negative/non-positive integer. The natural number following 0 is 1 and no natural number precedes 0. The number 0 may or may not be considered a natural number but it is a whole number and hence a rational number and real number (as well as algebraic number and a complex number)

The number 0 is neither positive nor negative and appears in the middle of a number line.

As a new approach, the following theory is propounded.,

THEORY "0" (zero) has either positive or negative character.

PROOF Let's discuss a simple division. Say 161 divided by 4

$$\begin{array}{r} 4\,\overline{)\,161\,}(40 \\ \underline{16} \\ 1 \\ \underline{0} \\ 1 \end{array}$$

Now, the remainder is 1 (one). Let us focus on this remainder. Generally, remainder occurs when something is taken out from something.

Mathematically, 'Remainder' occurs when a positive number is subtracted from a bigger positive number.

Then, 1 (one) cannot be called a remainder before subtracting 0 (zero) from 1 (one). But a question : why before or after subtracting 0 (zero) from 1 (one), 1 (one) itself exists arises. The answer to this question should be that 0 (zero) is a positive number, which is negligible. Hence, 0 (zero) has positive character.

Using "Euclid's Division Lemma" which states as under

Let a and b be any two integers and $b > 0$. Then, there exists unique integers q and r such that $a = bq + r$ and $0 \leq r < b$. Here, q is called the quotient of a with respect to b and r is the remainder of a with respect to b. Any such type of division may be discussed. Again, **Let us consider another case.**
We know that any positive number divided by zero is undefined.

Say $\frac{2}{0} = \infty$

Now, let us prove $\frac{2}{0} = \infty$

PROOF We get $\frac{2}{0.1} = 20$

$$\frac{2}{0.01} = 200$$

$$\frac{2}{0.001} = 2000$$

$$\frac{2}{0.0001} = 20000$$

.....

$$\frac{2}{0.0001 \ldots\ldots 1} = 200 \ldots\ldots$$

$$\Rightarrow \quad \frac{2}{0} = 2000 \ldots\ldots = \infty$$

It shows that 0.000 ..... 1 = 0

Therefore, zero is a positive number which is negligible.

Hence, 0 (zero) has positive character. Also, we know that for any real number x having positive character, there exists number $(-x)$ having negative character. However, $0 = -0$; thereby 0 has negative character also.
Now, the concept, "0 is neither positive nor negative" is to be changed into "0 is either positive or negative".

Let us, see the two series
(i) 0, 1, 2, 3,..... (ii) – 0, – 1, – 2, – 3,.....

Disclaimer,

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TARGET JEE 2017

TEST RIDER

Comprehensive Simulator Test Series for JEE Main & Advanced

JEE MAIN

(PAPER ONE PREP UP)

HALF TEST 1

(XI SYLLABUS) FULLY SOLVED

Mock Questions from Class XIth Syllabus with Complete Solutions

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 marks will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. If the lines $x + 2y - 1 = 0$, $ax + y + 3 = 0$ and $bx - y + 2 = 0$ are concurrent and S is the curve denoting the locus of (a, b), then the least distance of S from the origin is

(a) $\frac{5}{\sqrt{57}}$ (b) $\frac{5}{\sqrt{51}}$ (c) $\frac{5}{\sqrt{58}}$ (d) $\frac{5}{\sqrt{59}}$

2. Let $f(x)$ be a polynomial of even degree satisfying $f(2x)\left(1 - f\left(\frac{1}{2x}\right)\right) + f(16x^2y) = f(-2) - f(4xy)$ for all $x, y \in R \sim \{0\}$ and $f(4) = -255$, $f(0) = 1$. Then, the value of $\left|\frac{f(2) + 1}{2}\right|$ is

(a) 4 (b) 5 (c) 7 (d) 6

3. If $x^n = a_0 + a_1(1 + x) + a_2(1 + x)^2 + \ldots + a_n(1 + x)^n$
$= b_0 + b_1(1 - x) + b_2(1 - x)^2 + \ldots + b_n(1 - x)^n$,
then for $n = 201$, (a_{101}, b_{101}) is equal to

(a) $(-{}^{201}C_{101}, -{}^{201}C_{101})$ (b) $({}^{201}C_{101}, -{}^{201}C_{101})$
(c) $(-{}^{201}C_{101}, {}^{201}C_{101})$ (d) $({}^{201}C_{101}, {}^{201}C_{101})$

4. The value of c for which the set $\{(x, y) \mid x^2 + y^2 + 2x \le 1\} \cap \{(x, y) \mid x - y + c \ge 0\}$ contains only one point in common is

(a) $(-\infty, -1] \cup [3, \infty)$ (b) $\{-1, 3\}$
(c) $\{-3\}$ (d) $\{-1\}$

5. Two mutually perpendicular tangents of the parabola $y^2 = 4ax$ meet the axis at P_1 and P_2. If S is the focus of the parabola, then $\frac{1}{SP_1} + \frac{1}{SP_2}$ is equal to

(a) $4/a$ (b) $2/a$ (c) $1/a$ (d) $1/4a$

6. Let P be any point on a directrix of an ellipse of eccentricity e, S be the corresponding focus and C be the centre of the ellipse. The line PC meets the ellipse at A. The angle between PS and tangent at A is α. Then, α is equal to

(a) $\tan^{-1} e$ (b) $\frac{\pi}{2}$
(c) $\tan^{-1}(1 - e^2)$ (d) None of these

7. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that $\min f(x) > \max g(x)$, then the relation between b and c is

(a) no relation (b) $0 < c < \frac{b}{2}$
(c) $|c| < \frac{|b|}{\sqrt{2}}$ (d) $|c| > \sqrt{2}|b|$

8. The remainder when $8^{2n} - 62^{2n+1}$ divided by 9 is

(a) 7 (b) 8
(c) 0 (d) 2

9. If the inequality $\sin^2 x + a\cos x + a^2 > 1 + \cos x$ holds for any $x \in R$, then the largest negative integral value of a is

(a) -1 (b) -2
(c) -3 (d) -4

10. If the sides a, b, c of a ΔABC from successive terms of GP with common ratio $r(> 1)$, then which of the following is correct?

(a) $A > \pi/3$ (b) $B \ge \pi/3$
(c) $C < \pi/3$ (d) $A < B < \pi/3 < C$

11. If $x\sin a + y\sin 2a + z\sin 3a = \sin 4a$,
$x\sin b + y\sin 2b + z\sin 3b = \sin 4b$,
$x\sin c + y\sin 2c + z\sin 3c = \sin 4c$,
then the roots of the equation
$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$; $a, b, c \neq n\pi$, are
(a) $\sin a, \sin b, \sin c$ *(b)* $\cos a, \cos b, \cos c$
(c) $\sin 2a, \sin 2b, \sin 2c$ *(d)* $\cos 2a, \cos 2b, \cos 2c$

12. The number of solutions of
$[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$ (where $[\cdot]$ denotes the greatest integer function), $x \in [0, 2\pi]$, is
(a) 0 *(b)* 4 *(c)* infinite *(d)* 1

13. Six points (x_i, y_i); $i = 1, 2, \ldots, 6$ are taken on the circle $x^2 + y^2 = 4$ such that $\sum_{i=1}^{6} x_i = 8$ and $\sum_{i=1}^{6} y_i = 4$.
The line segment joining the orthocentre of the triangle made by any three points and the centroid of the triangle made by other three points passes through a fixed point (h, k). Then, $(h + k)$ is
(a) 4 *(b)* 3 *(c)* 2 *(d)* 1

14. The number of irrational solutions of the equation $\sqrt{x^2 + \sqrt{x^2 + 11}} + \sqrt{x^2 - \sqrt{x^2 + 11}} = 4$ is
(a) 0 *(b)* 2 *(c)* 4 *(d)* infinite

15. A circle with centre at the origin and radius equal to a meets the X-axis at A and B. $P(\alpha)$ and $Q(\beta)$ are two points on this circle, so that $\alpha - \beta = 2\gamma$, where γ is a constant. The locus of the point of intersection of AP and BQ is
(a) $x^2 - y^2 - 2ay\tan\gamma = a^2$ *(b)* $x^2 + y^2 - 2ay\tan\gamma = a^2$
(c) $x^2 + y^2 + 2ay\tan\gamma = a^2$ *(d)* $x^2 - y^2 + 2ay\tan\gamma = a^2$

16. The domain of definition of the function $f(x) = \log_2(\log_{1/2}(x^2 + 4x + 3)) + \sin^{-1}(2[x]^2 - 3)$, where $[x]$ denotes the greatest integer $\leq x$, is
(a) $0 < x \leq 1$ *(b)* $0 \leq x \leq 1$
(c) $-\infty < x \leq 0$ *(d)* None of these

17. Sum of series $S = n^2\left(\frac{C_0}{C_1}\right) + (n-1)^2\left(\frac{C_1}{C_2}\right) + \ldots + 1^2\left(\frac{C_{n-1}}{C_n}\right)$ equals
(a) $\frac{1}{6}n(n+1)(2n+1)$ *(b)* $\frac{1}{3}n(n+1)(n+2)$
(c) $\frac{1}{6}n(n+1)(n+2)$ *(d)* $\frac{1}{6}n^2(n+1)(n+2)$

18. If x, y, z are positive real numbers such that $x^3y^2z^4 = 7$, then the least value of $2x + 5y + 3z$ is
(a) $\left(\frac{528}{128}\right)^{1/9}$ *(b)* $3\left(\frac{525}{128}\right)^{1/9}$
(c) $9\left(\frac{525}{128}\right)^{1/9}$ *(d)* None of these

19. Consider the following propositions
p : I take medicine, q : I can sleep.
Then, the compound statement $\sim p \rightarrow \sim q$ means
(a) If I do not take medicine, then I cannot sleep
(b) If I do not take medicine, then I can sleep
(c) I take medicine iff I can sleep
(d) I take medicine iff I can sleep

20. If a and b are the greatest values of ${}^{2n}C_r$ and ${}^{2n-1}C_r$ respectively, then
(a) $a = 2b$ *(b)* $b = 2a$ *(c)* $a = b$ *(d)* None of these

21. The mode of the following distribution is

Class interval	1-5	6-10	11-15	16-20	21-25
Frequency	4	7	10	8	6

(a) 14.5 *(b)* 16.5 *(c)* 10.5 *(d)* 13.5

22. If the sum of x terms of a series is $S_x = \frac{1}{(2x+3)(2x+1)}$ whose rth term is T_r. Then, $\sum_{r=1}^{n}\frac{1}{T_r}$ is equal to
(a) $\frac{1}{4}\Sigma(2r+1)(2r-1)(2r+3)$
(b) $-\frac{1}{4}\Sigma(2r+1)(2r-1)(2r+3)$
(c) $\Sigma(2r+1)(2r-1)(2r+3)$
(d) None of the above

23. If the standard deviation of the data 1, 3, 5, 7, …, 2017 is σ, then $[\sigma]$ is (where, $[\cdot]$ denotes the greatest integer function)
(a) 580 *(b)* 581 *(c)* 582 *(d)* 583

24. The sum of all the numbers of four different digits that can be made using the digits 0, 1, 2 and 3 is
(a) 26664 *(b)* 399996 *(c)* 38664 *(d)* None of these

25. If all real values of x obtained from the equation $4^x - (a-3)2^x + a - 4 = 0$ are non-positive, then a lies in
(a) $(4, 5]$ *(b)* $(0, 4)$ *(c)* $(4, \infty)$ *(d)* None of these

26. The number of solutions of the equation $z^3 + \bar{z} = 0$ is
(a) 2 *(b)* 3 *(c)* 4 *(d)* 5

27. If a^r is the coefficient of x^r in the expansion of $(1 + x + x^2)^n$, then $a_1 - 2a_2 + 3a_3 - \ldots - 2na_{2n}$ equals
(a) 0 *(b)* n *(c)* $-n$ *(d)* $2n$

28. If $x > 1$, then the least value of expression $2\log_{10} x - \log_x 0.01$ is
(a) 1 *(b)* 2 *(c)* 4 *(d)* None of these

29. Let A and B be two sets containing 4 and 2 elements, respectively. Then, the number of subsets of the set $A \times B$ each having atleast three elements is
(a) 275 *(b)* 510 *(c)* 215 *(d)* 219

30. If z_1, z_2, z_3 are the vertices of an isosceles triangle right angled at z_2, then
(a) $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$
(b) $z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$
(c) $z_1^2 + z_2^2 + 2z_3^2 = 2z_2(z_1 + z_3)$
(d) $2z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$

Answers with Explanation

1. (*c*) The given lines are concurrent, if

$$\begin{vmatrix} 1 & 2 & -1 \\ a & 1 & 3 \\ b & -1 & 2 \end{vmatrix} = 0 \quad \text{or} \quad 7b - 3a + 5 = 0$$

The locus of (a, b) is $3x - 7y = 5$.
Least distance from $(0, 0)$

$$= \text{Length of perpendicular from } (0, 0) = \frac{5}{\sqrt{58}}$$

2. (*c*) Replacing y by $\frac{1}{8x^2}$ in the given functional equation, we obtain

$$f(2x)\left(1 - f\left(\frac{1}{2x}\right)\right) + f(2) = f(-2) - f\left(\frac{1}{2x}\right)$$

Since, f is an even function, so $f(2) = f(-2)$

$$\therefore \quad f(2x) - f(2x)f\left(\frac{1}{2x}\right) = -f\left(\frac{1}{2x}\right)$$

$$\Rightarrow \quad f(2x) + f\left(\frac{1}{2x}\right) = f(2x)f\left(\frac{1}{2x}\right)$$

Replacing x by $\frac{x}{2}$, we have $f(x) + f\left(\frac{1}{x}\right) = f(x)f\left(\frac{1}{x}\right)$

Since, f is polynomial, so $f(x) = \pm x^n + 1$.
But $-255 = f(4) = \pm 4^n + 1$
Only negative sign is possible, thus $4^n = 256 \Rightarrow n = 4$
i.e. $f(x) = -x^4 + 1 \Rightarrow f(2) = -2^4 + 1 = -15$

$$\therefore \quad \left|\frac{f(2) + 1}{2}\right| = \left|\frac{-15 + 1}{2}\right| = 7$$

3. (*b*) $x^n = [(1 + x) - 1]^n = [1 - (1 - x)]^n$

$$= \sum_{k=0}^{n} {}^nC_k(1 + x)^{n-k}(-1)^k = \sum_{k=0}^{n} {}^nC_k(-1)^k(1 - x)^k$$

$\therefore$ a_k = Coefficient of $(1 + x)^k$ in

$$\sum_{k=0}^{n} {}^nC_k(1 + x)^{n-k}(-1)^k = (-1)^{n-k}\,{}^nC_{n-k} = (-1)^{n-k}\,{}^nC_k$$

and $\quad b_k = {}^nC_k(-1)^k$

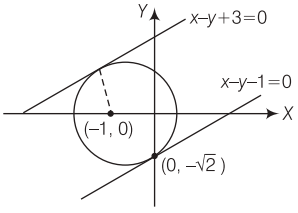
For $n = 201$, $k = 101$, we get

$$(a_{101}, b_{101}) = ({}^{201}C_{101}, -{}^{201}C_{101})$$

4. (*d*) $x^2 + y^2 + 2x - 1 = 0$

Centre $\equiv (-1, 0)$ and radius $= \sqrt{2}$

The line $x - y + c = 0$ must be tangent to the circle.



Therefore, $\left|\frac{-1 + c}{\sqrt{2}}\right| = \sqrt{2}$

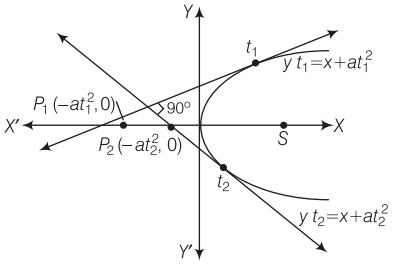
$\Rightarrow$ $|c - 1| = 2$

$\Rightarrow$ $c - 1 = \pm 2$

i.e. $c = 3$ or -1

$\Rightarrow$ $c = 1$ [$\because$ for $c = 3$, there will be infinite points common lying inside the circle]

5. (*c*) $SP_1 = a(1 + t_1^2)$, $\quad SP_2 = a(1 + t_2^2)$



$\therefore$ $t_1 t_2 = -1$

$$\frac{1}{SP_1} = \frac{1}{a(1 + t_1^2)}, \quad \frac{1}{SP_2} = \frac{1}{a(1 + t_2^2)} = \frac{t_1^2}{a(t_1^2 + 1)}$$

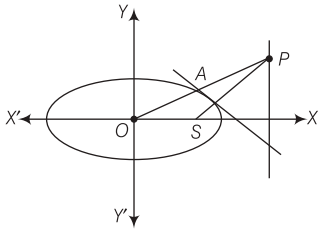
$$\therefore \quad \frac{1}{SP_1} + \frac{1}{SP_2} = \frac{1}{a}$$

6. (*b*) Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Again, let $A \equiv (a\cos\theta, b\sin\theta)$

The equation of AC will be $y = \frac{b}{a}\tan\theta x$.

On solving with $x = \frac{a}{e}$, we get $\quad P \equiv \left(\frac{a}{e}, \frac{b}{e}\tan\theta\right)$



$\therefore$ Slope of tangent at $A = -\frac{b}{a\tan\theta}$

and slope of PS

$$= \frac{\frac{b}{e}\tan\theta}{\frac{a}{e} - ae} = \frac{b\tan\theta}{a(1 - e^2)} = \frac{a}{b}\tan\theta$$

So, $\quad \alpha = \frac{\pi}{2}$

7. (*d*) $f(x) = (x + b)^2 + 2c^2 - b^2 \Rightarrow \min f(x) = 2c^2 - b^2$

Also, $\quad g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x + c)^2$

$\Rightarrow \max g(x) = b^2 + c^2$

As, $\min f(x) > \max g(x)$, we get

$$2c^2 - b^2 > b^2 + c^2 \Rightarrow c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}|b|$$

8. (*d*) $8^{2n} - 62^{2n+1} = (9 - 1)^{2n} - (63 - 1)^{2n+1}$

$$= ({}^{2n}C_0\,9^{2n} + {}^{2n}C_1\,9^{2n-1} + \ldots + {}^{2n}C_{2n}) - ({}^{2n+1}C_0\,63^{2n+1} + {}^{2n+1}C_1\,63^{2n} + {}^{2n+1}C_2\,63^{2n-1} + \ldots + {}^{2n+1}C_{2n+1})$$

$= 9m + 1 + 1 = 9m + 2$ for some integer m

Thus, when $8^{2n} - 62^{2n+1}$ is divided by 9, the remainder is 2.

9. (*c*) $\sin^2 x + a\cos x + a^2 > 1 + \cos x$

Putting $x = 0$, we have

$$a + a^2 > 2$$

or $\quad a^2 + a - 2 > 0$

or $\quad (a + 2)(a - 1) > 0$

or $\quad a < -2$ or $a > 1$

Therefore, the largest negative integral value of a is -3.

10. (*d*) Let the sides of the triangle be a, ar, ar^2.

$\because$ ar^2 is the longest side $(r > 1)$.

$\therefore$ $a + ar > ar^2$

So, $\cos C = \dfrac{a^2 + a^2r^2 - a^2r^4}{2a^2r} = \dfrac{1 + r^2 - r^4}{2r} < \dfrac{1}{2} \Rightarrow C > \dfrac{\pi}{3}$

Also, $\cos B = \dfrac{a^2 + a^2r^4 - a^2r^2}{2a^2r^2} = \dfrac{1 + r^4 - r^2}{2r^2}$

$= \dfrac{1}{2}\left[r^2 + \dfrac{1}{r^2} - 1\right] = \dfrac{1}{2}\left[\left(r - \dfrac{1}{r}\right)^2 + 1\right] > \dfrac{1}{2} \Rightarrow B < \dfrac{\pi}{3}$

$\therefore$ $a < ar < ar^2 \Rightarrow A < B < C \Rightarrow A < B < \dfrac{\pi}{3} < C$

11. (*b*) $x \sin \alpha + y \sin 2\alpha + z \sin 3\alpha = \sin 4\alpha$

$\Rightarrow$ $x \sin \alpha + y \times 2\sin \alpha \cos \alpha + z \times \sin \alpha(3 - 4\sin^2 \alpha)$

$= 2 \times 2\sin \alpha \cos \alpha \cos 2\alpha$

$\Rightarrow$ $x + 2y \cos \alpha + z(3 + 4\cos^2 \alpha - 4)$

$= 4\cos \alpha(2\cos^2 \alpha - 1)$ [as $\sin \alpha \neq 0$]

$\Rightarrow$ $\cos^3 \alpha - \left(\dfrac{z}{2}\right)\cos^2 \alpha - \left(\dfrac{y + 2}{4}\right)\cos \alpha + \left(\dfrac{z - x}{8}\right) = 0$

which shows that $\cos \alpha$ is root of the equation

$$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y + 2}{4}\right)t + \left(\frac{z - x}{8}\right) = 0.$$

Similarly, from second and third equations, we can show that $\cos a$, $\cos b$ and $\cos c$ are the roots of the given equation.

12. (*c*) Given, $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$

The maximum value of the LHS is 1 and the minimum of the RHS is also 1.

Thus, $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x] = 1$

or $[\sin x + \cos x] = 1, [-\sin x] = -1, [-\cos x] = -1$

$\therefore$ $1 \leq (\sin x + \cos x) \leq \sqrt{2}$

$\therefore$ $x \in \left[0, \dfrac{\pi}{2}\right]$ ...(i)

$[-\sin x] = -1$

$\therefore$ $-1 \leq (-\sin x) < 0$

$\therefore$ $0 < \sin x \leq 1$

$\therefore$ $x \in \left(0, \dfrac{\pi}{2}\right]$ ...(ii)

$[-\cos x] = -1$

$\therefore$ $-1 \leq -\cos x < 0$

$\therefore$ $0 < \cos x \leq 1$

$\therefore$ $x \in \left[0, \dfrac{\pi}{2}\right)$ ...(iii)

From Eqs. (i), (ii) and (iii), $x \in \left(0, \dfrac{\pi}{2}\right)$

13. (*b*) Let $\sum_{i=1}^{6} x_i = \alpha$ and $\sum_{i=1}^{6} y_i = \beta$

Let O be the orthocentre of the triangle made by $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3).

Then, $O \equiv (x_1 + x_2 + x_3, y_1 + y_2 + y_3) \equiv (\alpha_1, \beta_1)$

Similarly, let G be the centroid of the triangle made by other three points. Then,

$G \equiv \left(\dfrac{x_4 + x_5 + x_6}{3}, \dfrac{y_4 + y_5 + y_6}{3}\right) \equiv \left(\dfrac{\alpha - \alpha_1}{3}, \dfrac{\beta - \beta_1}{3}\right)$

The point dividing OG in the ratio 3 : 1 is $\left(\dfrac{\alpha}{4}, \dfrac{\beta}{4}\right) \equiv (2, 1)$

Therefore, $h + k = 3$

14. (*b*) Put $\sqrt{x^2 + 11} = t$

$\Rightarrow x^2 + 11 = t^2$, so that equation becomes

$$\sqrt{t^2 + t - 11} + \sqrt{t^2 - t - 11} = 4$$

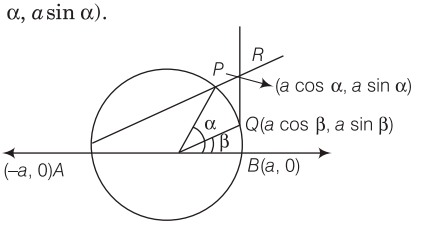
Since, $(t^2 + t - 11) - (t^2 - t - 11) \equiv 2t$, we get

$$\sqrt{t^2 + t - 11} - \sqrt{t^2 - t - 11} = \frac{t}{2}$$

$\therefore$ $2\sqrt{t^2 + t - 11} = 4 + \dfrac{t}{2} \Rightarrow 4(t^2 + t - 11) = 16 + \dfrac{1}{4}t^2 + 4t$

$\Rightarrow$ $t^2 = 16 \Rightarrow x^2 + 11 = 16 \Rightarrow x = \pm\sqrt{5}$

15. (*b*) Coordinates of A are $(-a, 0)$ and P are $(a \cos \alpha, a \sin \alpha)$.



$\therefore$ Equation of AP is

$$y = \frac{a \sin \alpha}{a(\cos \alpha + 1)}(x + a) \text{ or } y = \tan\left(\frac{\alpha}{2}\right)(x + a) \quad ...(i)$$

Similarly, equation of BQ is

$$y = \frac{a \sin \beta}{a \cos (\beta - 1)}(x - a) \text{ or } y = -\cot\left(\frac{\beta}{2}\right)(x - a) \quad ...(ii)$$

We now eliminate α, β from Eqs. (i) and (ii).

From Eqs. (i) and (ii), $\tan\left(\dfrac{\alpha}{2}\right) = \dfrac{y}{a + x}$, $\tan\left(\dfrac{\beta}{2}\right) = \dfrac{a - x}{y}$

Now, $\alpha - \beta = 2\gamma$

$$\Rightarrow \tan \gamma = \frac{\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right)}{1 + \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)} = \frac{\frac{y}{a + x} - \frac{a - x}{y}}{1 + \frac{y}{a + x} \cdot \frac{a - x}{y}}$$

$$\Rightarrow \tan \gamma = \frac{y^2 - (a^2 - x^2)}{(a + x)y + (a - x)y} = \frac{x^2 + y^2 - a^2}{2ay}$$

$$\Rightarrow x^2 + y^2 - 2ay \tan \gamma = a^2$$

which is the required locus.

16. (*d*) The function

$f(x) = \log_2(\log_{1/2}(x^2 + 4x + 3)) + \sin^{-1}[x]$ is defined, if

$0 < x^2 + 4x + 3 < 1$ and $-1 \leq x \leq 2$

$x^2 + 4x + 3 > 0 \Rightarrow x > -1$ or $x < -3$

$x^2 + 4x + 3 < 1 \Rightarrow -2 - \sqrt{2} < x < -2 + \sqrt{2}$

So, the domain of the function is $-1 < x < -2 + \sqrt{2}$.

17. (*c*) As $\dfrac{C_k}{C_{k+1}} = \dfrac{k + 1}{n - k}$

$$S = n^2\left(\frac{1}{n}\right) + (n - 1)^2\left(\frac{2}{n - 1}\right) + \ldots + 1^2\left(\frac{n + 0}{1}\right)$$

$$= \sum_{k=0}^{n}(n - k)(k + 1) = n\sum_{k=0}^{n}(k + 1) - \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k$$

$$= n\left(\frac{1}{2}\right)(n + 1)(n + 2) - \left(\frac{1}{6}\right)n(n + 1)(2n + 1) - \left(\frac{1}{2}\right)n(n + 1)$$

$$= \frac{1}{6}n(n + 1)(n + 2)$$

18. (*c*) Let $S = 2x + 5y + 3z$

Since, $x^3y^2z^4 = 7$ is the product of 9 terms *viz.* $3x$'s, $2y$'s and $4z$'s. So, we write S as the sum of 9 terms as given below.

$\therefore \quad S = 3\left(\frac{2x}{3}\right) + 2\left(\frac{5y}{2}\right) + 4\left(\frac{3z}{4}\right)$

Now, $P = \left(\frac{2x}{3}\right)^3\left(\frac{5y}{2}\right)^2\left(\frac{3z}{4}\right)^4$

$= \left(\frac{2}{3}\right)^3\left(\frac{5}{2}\right)^2\left(\frac{3}{4}\right)^4 (x^3y^2z^4) = \left(\frac{2}{3}\right)^3\left(\frac{5}{2}\right)^2\left(\frac{3}{4}\right)^4 \times 7$

$= \frac{525}{128}$ (Constant)

Therefore, S is least when $\frac{2x}{3} = \frac{5y}{2} = \frac{3z}{4} = \lambda$ (say)

$\Rightarrow \quad x = \frac{3\lambda}{2}, y = \frac{2\lambda}{5}$ and $z = \frac{4\lambda}{3}$

Given, $x^3y^2z^4 = 7 \Rightarrow \lambda^9 = \left(\frac{525}{128}\right) \Rightarrow \lambda = \left(\frac{525}{128}\right)^{1/9}$

Then, the least value of S is given by

$$S = 2 \times \frac{3\lambda}{2} + 5\left(\frac{2\lambda}{5}\right) + 3\left(\frac{4\lambda}{3}\right) = 9\lambda = 9\left(\frac{525}{128}\right)^{1/9}$$

19. (*a*) We have,

$\sim p$: I do not take medicine, $\sim q$: I cannot sleep

$\therefore \sim p \rightarrow \sim q$: If I do not take medicine, then I cannot sleep.

20. (*a*) We know that, the greatest value of ${}^nC_r$ is given by

$$\begin{cases} {}^nC_{n/2}, & \text{if } n \text{ is even} \\ {}^nC_{(n-1)/2} \text{ or } {}^nC_{(n+1)/2}, & \text{if } n \text{ is odd} \end{cases}$$

$\therefore \quad a =$ Greatest value of ${}^{2n}C_r = {}^{2n}C_n$

and $b =$ Greatest value of ${}^{2n-1}C_r = {}^{2n-1}C_{n-1}$

$$\Rightarrow \frac{a}{b} = \frac{{}^{2n}C_n}{{}^{2n-1}C_{n-1}} = \frac{\frac{2n}{n}{}^{2n-1}C_{n-1}}{{}^{2n-1}C_{n-1}} = 2 \Rightarrow a = 2b$$

21. (*d*) The given distribution is not a continuous frequency distribution. So, we first convert it into a continuous frequency distribution as given below

Class interval	0.5-5.5	5.5-10.5	10.5-15.5	15.5-20.5	20.5-25.5
Frequency	4	7	10	8	6

Clearly, 10.5-15.5 is the modal class such that

$l = 10.5, f = 10, f_1 = 7, f_2 = 8$ and $h = 5$

$\therefore$ Mode $= l + \frac{f - f_1}{2f - f_1 - f_2} \times h = 10.5 + \frac{3}{5} \times 5 = 13.5$

22. (*b*) We have, $S_x = \frac{1}{(2x+3)(2x+1)}$

$\therefore \quad T_x = S_x - S_{x-1}$

$\Rightarrow \quad T_x = \frac{1}{(2x+3)(2x+1)} - \frac{1}{(2x+1)(2x-1)}$

$\Rightarrow \quad T_x = \frac{-4}{(2x-1)(2x+1)(2x+3)}$

$\therefore \sum_{r=1}^{n} \frac{1}{T_r} = \frac{-1}{4}\sum_{r=1}^{n}(2r-1)(2r+1)(2r+3)$

23. (*c*) Let $\bar{X}$ be the mean of the given data. Then,

$$\bar{X} = \frac{1 + 3 + 5 + 7 + \ldots + 2017}{1009} = \frac{(1009)^2}{1009} = 1009$$

$\therefore$ Variance $(\sigma^2) = \frac{1}{1009}(1^2 + 3^2 + \ldots + 2017^2) - (1009)^2$

$\Rightarrow \quad \sigma^2 = 339360$

$\therefore \quad \sigma = 582.54$

Hence, $[\sigma] = 582$

24. (*c*) We have,

Required sum = (Sum of all the numbers formed by the digits 0, 1, 2, 3) – (Sum of all the numbers having 0 at thousand place)

$$= (0 + 1 + 2 + 3) \times (4-1)!\left(\frac{10^4 - 1}{10 - 1}\right) - (1 + 2 + 3)(3-1)!\left(\frac{10^3 - 1}{10 - 1}\right)$$

$$= 6 \times 3!\left(\frac{10^4 - 1}{9}\right) - 6 \times 2!\left(\frac{10^3 - 1}{9}\right)$$

$$= 4(10^4 - 1) - \frac{4}{3} \times 999 = 40000 - 4 - 1332 = 38664$$

25. (*a*) We have, $4^x - (a-3)2^x + a - 4 = 0$

$\Rightarrow \quad (2^x)^2 - (a-3)2^x + a - 4 = 0$

$\Rightarrow \quad (2^x)^2 - (a-4)2^x - 2^x + a - 4 = 0$

$\Rightarrow \quad 2^x[2^x - (a-4)] - [2^x - (a-4)] = 0$

$\Rightarrow \quad (2^x - 1)[2^x - (a-4)] = 0$

$\Rightarrow \quad 2^x = 1, a = -4$

$x \le 0 \Rightarrow 0 < 2^x \le 1 \Rightarrow 0 < a - 4 \le 1$

$\Rightarrow \quad 4 < a \le 5 \Rightarrow a \in (4, 5]$

26. (*d*) We have, $z^3 + \bar{z} = 0 \Rightarrow z^3 = -\bar{z}$

$\Rightarrow \quad |z|^3 = |\bar{z}| \Rightarrow |z|(|z|^2 - 1) = 0$

$\Rightarrow \quad |z| = 0$ or $|z| = 1$

If $|z| = 0$, then $z = 0$

If $|z| = 1$, then $|z|^2 = 1 \Rightarrow z\bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z}$

$\therefore \quad z^3 + \bar{z} = 0 \Rightarrow z^3 + \frac{1}{z} = 0 \Rightarrow z^4 + 1 = 0$

Clearly, $z^4 + 1 = 0$ has four non-zero and distinct roots. So, the given equation has five roots.

27. (*c*) We have, $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$

On differentiating both sides w.r.t. x, we get

$n(1 + x + x^2)^{n-1}(1 + 2x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + 2na_{2n}x^{2n-1}$

Putting $x = -1$ on both sides, we get

$-n = a_1 - 2a_2 + 3a_3 - \ldots - 2na_{2n}$

28. (*c*) We have, $2\log_{10}x - \log_x 0.01 = 2\log_{10}x - \log_x(10)^{-2}$

$$= 2\log_{10}x + 2\log_x 10 = 2\left(\log_{10}x + \frac{1}{\log_{10}x}\right)$$

$\ge 2 \times 2 = 4$ $\quad [\because x > 1 \Rightarrow \log_{10}x > 0]$

29. (*d*) We have, $n(A) = 4$ and $n(B) = 2$

$\therefore \quad n(A \times B) = n(A) \times n(B) = 4 \times 2 = 8$

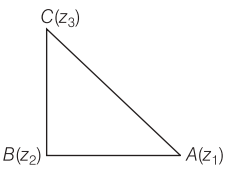
Required number of subsets

$$= {}^8C_3 + {}^8C_4 + {}^8C_5 + \ldots + {}^8C_8 = \left(\sum_{r=0}^{8} {}^8C_r\right) - ({}^8C_0 + {}^8C_1 + {}^8C_2)$$

$= 2^8 - (1 + 8 + 28) = 256 - 37 = 219$

30. (*a*) We have,

$AB = BC$ and $\angle ABC = \frac{\pi}{2}$



$\therefore \quad z_3 - z_2 = (z_1 - z_2)e^{\pm i\pi/2}$

$\Rightarrow \quad (z_3 - z_2) = \pm i(z_1 - z_2)$

$\Rightarrow \quad (z_3 - z_2)^2 = -(z_1 - z_2)^2$

$\Rightarrow \quad z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$

TARGET JEE 2017

TEST RIDER

Comprehensive Simulator Test Series for JEE Main & Advanced

JEE MAIN

(PAPER TWO SCALE UP)

HALF TEST 1

(XI SYLLABUS)

WITH ANSWER KEY

Mock Questions from Class XIth syllabus to inhance Your Problem Solving Skills

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 marks will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. If the product of the roots of the equation $x^2 - 2\sqrt{2}kx + 2^{2\log k} - 1 = 0$ is 31. Then, the roots of the equation are real for k is equal to

(a) 1 *(b)* 2 *(c)* 3 *(d)* 4

2. How many ways are there to arrange the letters in the word 'GARDEN' with vowels in alphabetical order?

(a) 480 *(b)* 240 *(c)* 360 *(d)* 120

3. Let $f(x) = \sum_{r=1}^{n} r^2 C_r^2$, then $f(5)$ is equal to

(a) ${}^8C_4$ *(b)* $25 \times {}^8C_4$ *(c)* $25 \times {}^8C_5$ *(d)* None of these

4. Given $A(0, 0)$ and $B(x, y)$ with $x \in (0, 1)$ and $y > 0$. Let the slope of line AB be m_1, point C lies on line $x = 1$ such that the slope of BC is equal to m_2, where $0 < m_2 < m_1$. If the area of ΔABC can be expressed as $(m_1 - m_2)f(x)$, then the largest possible value of x is

(a) 1 *(b)* 1/2 *(c)* 1/4 *(d)* 1/8

5. Let A_r, $r = 1, 2, 3, \ldots$ be the points on the number line such that $OA_1, OA_2, OA_3, \ldots$ are in GP, where O is the origin and the common ratio of the GP be a positive proper fraction. Let M_r be the middle point of the line segment A_rA_{r+1}. Then, the value of $\sum_{r=1}^{\infty} OM_r$ is equal to

(a) $\dfrac{OA_2(OA_1 - OA_2)}{2(OA_1 + OA_2)}$ *(b)* $\dfrac{OA_1(OA_1 - OA_2)}{2(OA_1 + OA_2)}$

(c) $\dfrac{OA_1}{2(OA_1 - OA_2)}$ *(d)* ∞

6. Number of solutions of $\theta \in [0, 2\pi]$ satisfying the equation $(\log_{\sqrt{3}} \tan\theta)(\sqrt{\log_{\tan\theta} 3 + \log_{\sqrt{3}} 3\sqrt{3}}) = -1$

(a) 1 *(b)* 2 *(c)* 3 *(d)* 4

7. If the equation of the pair of straight lines passing through the point (1, 1) are making an angle θ with the positive direction of the X-axis and the other making the same angle with the positive direction of the Y-axis, is $x^2 - (a + 2)xy + y^2 + a(x + y - 1) = 0$, $a \neq -2$, then the value of $\sin 2\theta$ is

(a) $a - 2$ *(b)* $a + 2$ *(c)* $2/a + 2$ *(d)* $2/a$

8. In a circle of radius r, chords of length a and b cm subtend angles θ and 3θ respectively at the centre. Then, the radius of circle is

(a) $a\sqrt{\dfrac{a}{3a - b}}$ *(b)* $\sqrt{\dfrac{a}{3a - b}}$ *(c)* a *(d)* None of these

9. If $\dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x} = 1$, then the smallest positive value of x is

(a) 60° *(b)* 50° *(c)* 120° *(d)* 45°

10. A tower AB leans towards West making an angle α with the vertical. The angular elevation of B, the top most point of the tower is β, as observed from a point C due East of A at a distance d from A. If the angular elevation of B from α point due East of C is at a distance $2d$ from C is γ, then $2\tan\alpha$ is equal to

(a) $\cot\beta - \cot\gamma$ *(b)* $\cot\beta - 3\cot\gamma$

(c) $3\cot\beta - \cot\gamma$ *(d)* $3\cot\beta - 3\cot\gamma$

11. ABC is an isosceles triangle in which A is $(-1, 0)$, $\angle A = 2\pi/3$, $AB = AC$ and AB is along the X-axis. If $BC = 4\sqrt{3}$, then the equation of the line BC is

(a) $x + \sqrt{3}y = 3$ *(b)* $\sqrt{3}x + y = 3$

(c) $x + y = \sqrt{3}$ *(d)* None of these

12. If two chords drawn from the point $A(4, 4)$ to the parabola $x^2 = 4y$ are bisected by the line $y = mx$, the interval in which m lies is

(a) $(-2\sqrt{2}, 2\sqrt{2})$ *(b)* $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(c) $(-\infty, -2\sqrt{2} - 2) \cup (2\sqrt{2} - 2, \infty)$ *(d)* None of these

13. Consider a square $ABCD$ of side length 1. Let P be the set of all segments of length 1 with end points on the adjacent sides of square $ABCD$. The mid-points of segments in P enclose a region with area A. The value of A is

(a) $\pi/4$ *(b)* $1-\pi/4$ *(c)* $4-\pi/4$ *(d)* None of these

14. $ABCD$ is a rhombus in the argand plane. If the affixes of the vertices are z_1, z_2, z_3 and z_4 respectively and $\angle CBA = \frac{\pi}{3}$, then

(a) $z_1 + \omega z_2 + \omega^2 z_3 = 0$ *(b)* $z_1 - \omega z_2 - \omega^2 z_3 = 0$
(c) $\omega z_1 + z_2 + \omega^2 z_3 = 0$ *(d)* $\omega^2 z_1 + \omega z_2 + z_3 = 0$

15. If $\left|z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is

(a) $\sqrt{5}$ *(b)* $\sqrt{5}+1$ *(c)* $\sqrt{5}-1$ *(d)* None of these

16. Tangent is drawn at any point (p, q) on the parabola $y^2 = 4ax$. Tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that the chords of contact pass through a fixed point (r, s). Then, p, q, r and s can hold the relation

(a) $r^2q = 4p^2s$ *(b)* $rq^2 = 4ps^2$
(c) $rq^2 = -4ps^2$ *(d)* $r^2q = -4p^2s$

17. The exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2 + \alpha^2 y^2 = \alpha^2$ and the portion of the tangent intercepted by the hyperbola $\alpha^2 x^2 - y^2 = 1$ subtends a right angle at the centre of the curves is

(a) $\left[\frac{\sqrt{5}+1}{2}, 2\right]$ *(b)* $(1, 2]$
(c) $\left[\frac{\sqrt{5}-1}{2}, 1\right)$ *(d)* $\left[\frac{\sqrt{5}-1}{2}, 1\right) \cup \left(1, \frac{\sqrt{5}+1}{2}\right]$

18. If the ellipse $\frac{x^2}{a^2-7} + \frac{y^2}{13-5a} = 1$ is inscribed in a square of side length $\sqrt{2}a$, then a is equal to

(a) 6/5 *(b)* $(-\infty, -\sqrt{7}) \cup (\sqrt{7}, 13/5)$
(c) $(-\infty, -\sqrt{7}) \cup (13/5, \sqrt{7})$ *(d)* No such a exists

19. If S denotes the set of all real x for which $(x^2 + x + 1)^x < 1$, then S is equal to

(a) $(1, \infty)$ *(b)* $(-1, \infty)$ *(c)* $(-\infty, -1)$ *(d)* None of these

20. Let A and B be two non-empty sets having n elements in common. Then, the number of elements common to $A \times B$ and $B \times A$ is

(a) $2n$ *(b)* n *(c)* n^2 *(d)* None of these

21. The set of values of m for which it is possible to drawn the chord $y = \sqrt{m}\,x + 1$ to the curve $x^2 + 2xy + (2 + \sin^2 x)y^2 = 1$, which subtends a right angle at the origin for some value of α, is

(a) $[2, 3]$ *(b)* $[0, 1]$ *(c)* $[1, 3]$ *(d)* None of these

22. If $\sum_{k=1}^{n}\left(\sum_{m=1}^{k} m^2\right) = an^4 + bn^3 + cn^2 + dn + e$, then

(a) $a = 1/12$ *(b)* $b = 1/6$ *(c)* $d = 1/4$ *(d)* $e = 0$

23. If ω is a non-real cube root of unity, x is a real number and $n \in N$ such that first three terms in the binomial expansion of $(\omega + x)^n$ are 1, $12\bar{\omega}$ and $69\,\omega$ respectively, then

(a) $n = 36, x = 1$ *(b)* $n = 12, x = 2$
(c) $n = 24, x = 1/2$ *(d)* $n = 18, x = 1/3$

24. If $A = \left\{\theta : \cos\theta > -\frac{1}{2}, 0 \le \theta \le \pi\right\}$ and $B = \left\{\theta : \sin\theta > \frac{1}{2}, \frac{\pi}{3} \le \theta \le \pi\right\}$, then

(a) $A \cap B = \left\{\theta : \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\right\}$

(b) $A \cap B = \left\{\theta : -\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\right\}$

(c) $A \cup B = \left\{\theta : -\frac{5\pi}{6} \le \theta \le \frac{5\pi}{6}\right\}$

(d) $A \cup B = \left\{\theta : 0 \le \theta \le \frac{\pi}{6}\right\}$

25. If $x + y + z = 1$, then the minimum value of $xy(x+y)^2 + yz(y+z)^2 + zx(z+x)^2$ is (where , $x, y, z \in R^+$)

(a) $4xyz$ *(b)* $3xyz$ *(c)* $2xyz$ *(d)* $6xyz$

26. For $(2n + 1)$ observations $x_1, -x_1, x_2, -x_2, \ldots, x_n, -x_n$ and 0, where x_i^s distinct non-zero numbers. Let SD and MD denote the standard deviation and median respectively. Then, which one of the following is always true?

(a) SD < MD *(b)* SD > MD *(c)* SD = MD
(d) Nothing can be said in general about the relationship of SD and MD

27. If $\log_{\cos x}\tan x + \log_{\sin x}\cot x = 0$, then x is equal to

(a) $n\pi + \frac{\pi}{4}, n \in Z$ *(b)* $2n\pi + \frac{\pi}{4}, n \in Z$
(c) $2n\pi - \frac{3\pi}{4}, n \in Z$ *(d)* None of these

28. If $x = 5$ and $y = -2$, then $x - 2y = 9$. Then, the contrapositive of this proposition is

(a) If $x - 2y = 9$, then $x \ne 5$ or $y \ne -2$
(b) If $x - 2y = 9$, $x \ne 5$ and $y \ne -2$
(c) $x - 2y = 9$ if and only if $x = 5$ and $y = -2$
(d) None of the above

29. If $f : [-2, 2] \to R$ is defined by

$$f(x) = \begin{cases} -1, & \text{for } -2 \le x \le 0 \\ x - 1, & \text{for } 0 \le x \le 2 \end{cases},$$

then $\{x \in [-2, 2] : x \le 0 \text{ and } f(|x|) = x\}$ is equal to

(a) $\{-1\}$ *(b)* $\{0\}$ *(c)* $\left\{-\frac{1}{2}\right\}$ *(d)* ϕ

30. If $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$, $z = \sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta$, $|\cos\theta| < 1, |\sin\theta| < 1$, then $x + y + z$ is equal to

(a) xy *(b)* yz *(c)* zx *(d)* xyz

Answers

1. (d)	2. (c)	3. (b)	4. (d)	5. (b)
6. (b)	7. (c)	8. (a)	9. (a)	10. (c)
11. (a)	12. (c)	13. (b)	14. (a)	15. (b)
16. (c)	17. (a)	18. (d)	19. (c)	20. (c)
21. (a)	22. (a)	23. (c)	24. (a)	25. (a)
26. (b)	27. (b)	28. (a)	29. (c)	30. (d)

MATRICES & DETERMINANTS

A collection of best questions asked in JEE to revise your concepts

Single Option Correct Type

1. Let $\omega\ (\neq 1)$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a, b and c is either ω or ω^2. Then, the number of distinct matrices in the set S is [2013]

(a) 2 *(b)* 6 *(c)* 4 *(d)* 8

2. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$. The values of c and d are [2005]

(a) $(-6, -11)$ *(b)* $(6, 11)$ *(c)* $(-6, 11)$ *(d)* $(6, -11)$

3. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is [2012]

(a) 2^{10} *(b)* 2^{11} *(c)* 2^{12} *(d)* 2^{13}

4. If $S_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $k \in N$, where N is the set of all natural numbers, then $(S_2)^n\ (S_k)^{-1}$ for $n \in N$, is [2014]

(a) S_{2n-k} *(b)* S_{2n+k-1} *(c)* S_{2^n+k-1} *(d)* S_{2^n+k+1}

5. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If $\det(\text{adj } A) + \det(\text{adj } B) = 10^6$, then $[k]$ is equal to [2010]

(a) 3 *(b)* 4 *(c)* 5 *(d)* 6

6. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then $\sin^2 A + \sin^2 B + \sin^2 C$ is [2014]

(a) $\frac{3\sqrt{3}}{2}$ *(b)* $\frac{9}{4}$ *(c)* $\frac{5}{4}$ *(d)* 2

7. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2\ (M^TN)^{-1}\ (MN^{-1})^T$ is equal to [2011]

(a) M^2 *(b)* $-N^2$ *(c)* $-M^2$ *(d)* MN

8. Let M be a 3×3 matrix satisfying

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix},\ M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}.$$

Then, the sum of the diagonal entries of M is [2011]

(a) 7 *(b)* 8 *(c)* 9 *(d)* 6

9. The number of values of k for which the system of equations $(k+1)x + 8y = 4k$, $kx + (k+3)y = 3k - 1$ has no solution, is [2013]

(a) infinite *(b)* 1 *(c)* 2 *(d)* 3

10. If A is an 3×3 non-singular matrix such that $AA^T = A^TA$ and $B = A^{-1}A^T$, then BB^T equals [2014]

(a) B^{-1} *(b)* $(B^{-1})^T$ *(c)* $I + B$ *(d)* I

Integer Type

11. Let ω be the complex number of $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then, the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to ..... . [2010]

12. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and U_1, U_2, U_3 be column matrix satisfying $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. If U is 3×3 matrix, whose columns are U_1, U_2, U_3, then $|U|$ is ..... . [2011]

13. Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1 such that five of these entries are 1 and four are 0. The number of matrix in set A is ..... . [2009]

14. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$ is ..... . [2009]

Multiple Options Correct Type

15. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then, which of the following matrices is(are) skew-symmetric? [2015]

(a) $Y^3Z^4 - Z^4Y^3$
(b) $X^{44} + Y^{44}$
(c) $X^4Z^3 - Z^3X^4$
(d) $X^{23} + Y^{23}$

16. Which of the following values of α satisfy the equation $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$? [2015]

(a) -4 (b) 9
(c) -9 (d) 4

Linked Comprehension Type

Let a, b and c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]. \quad [2011]$$

17. If the point $P(a, b, c)$ with reference to E lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is
(a) 0 (b) 12
(c) 7 (d) 6

18. Let w be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying E, then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{1}{\omega^c}$ is equal to
(a) -2 (b) 2 (c) 3 (d) -3

19. Let $b = 6$ with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is
(a) 6 (b) 7 (c) $\frac{6}{7}$ (d) ∞

Matrix Match Type

20. Consider the following linear equations

$$ax + by + cz = 0,$$
$$bx + cy + az = 0$$
and
$$cx + ay + bz = 0.$$

Match the expressions/statements in Column I with expressions statements in Column II. [2007]

Column I	Column II
A. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	p. the equations represent planes meeting only at a single point.
B. $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	q. the equations represent the line $x = y = z$.
C. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	r. the equations represent identical planes.
D. $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	s. the equations represent the whole of the three dimensional space.

Answers with Explanation

1. **(a)** For being non-singular, $\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$

or $ac\omega^2 - (a + c)\omega + 1 \neq 0$

Hence, the number of possible triplets of (a, b, c) is 2 i.e. $(\omega, \omega^2, \omega)$ and (ω, ω, ω).

2. **(c)** We have,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{6}(A^2 + cA + dI)$$

$$\Rightarrow 6AA^{-1} = A^3 + cA^2 + dAI$$

$$\Rightarrow A^3 + cA^2 + dA - 6I = 0$$

We find out

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}; A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix}$$

$$\therefore A^3 + cA^2 + dA - 6I = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix} + c\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + d\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$-6\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=0$$

$$\Rightarrow\begin{bmatrix}1+c+d-6 & 0 & 0\\ 0 & -11-c+d-6 & 19+5c+d\\ 0 & -38-10c-2d & 46+14c+4d-6\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\Rightarrow \quad 1+c+d-6=0$

$-11-c+d-6=0$

$\Rightarrow \quad c+d=5$ and $-c+d=17$

On solving, we get $c=-6, d=11$

3. ***(d)*** We have,

$Q=[b_{ij}]$, where $b_{ij}=2^{i+j}a_{ij}$

$\therefore \quad Q=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}$

$\Rightarrow \quad |Q|=\begin{vmatrix}2^2a_{11}&2^3a_{12}&2^4a_{13}\\2^3a_{21}&2^4a_{22}&2^5a_{23}\\2^4a_{31}&2^5a_{32}&2^6a_{33}\end{vmatrix}$

$\Rightarrow \quad |Q|=2^2\times2^3\times2^4\begin{vmatrix}a_{11}&2a_{12}&2^2a_{13}\\a_{21}&2a_{22}&2^2a_{23}\\a_{31}&2a_{32}&2^2a_{33}\end{vmatrix}$

$\Rightarrow \quad |Q|=2^9\times2\times2^2\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=2^{12}|P|$

$=2^{12}\times2=2^{13}$

4. ***(a)*** We have, $S_k=\begin{bmatrix}1&k\\0&1\end{bmatrix}$

$\therefore \quad (S_k)^{-1}=\begin{bmatrix}1&-k\\0&1\end{bmatrix}$ and $S_2=\begin{bmatrix}1&2\\0&1\end{bmatrix}$

Now, $S_2=\begin{bmatrix}1&2\\0&1\end{bmatrix}$

$\Rightarrow \quad (S_2)^2=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}=\begin{bmatrix}1&2\times2\\0&1\end{bmatrix}$

$\Rightarrow \quad (S_2)^3=(S_2)^2\,S_2=\begin{bmatrix}1&2\times2\\0&1\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}=\begin{bmatrix}1&2\times3\\0&1\end{bmatrix}$

In general, $(S_2)^n=\begin{bmatrix}1&2n\\0&1\end{bmatrix}$

$\therefore \quad (S_2)^n\,(S_k)^{-1}=\begin{bmatrix}1&2n\\0&1\end{bmatrix}\begin{bmatrix}1&-k\\0&1\end{bmatrix}=\begin{bmatrix}1&2n-k\\0&1\end{bmatrix}$

$=S_{2n-k}$

5. ***(b)*** Since B is a skew-symmetric matrix of odd order.

Therefore, $|B|=0$. Also, $|A|=(2k+1)^3$

$\therefore \det(\text{adj } A)+\det(\text{adj } B)=10^6$

$\Rightarrow \quad |A|^2+|B|^2=10^6$

$[\because |\text{adj } A|=|A|^{n-1}$ for $n\times n$ matrix $A]$

$\Rightarrow \quad (2k+1)^6=10^6\Rightarrow k=\dfrac{9}{2}$

Hence, $[k]=4$

6. ***(b)*** We have,

$$\begin{vmatrix}1&a&b\\1&c&a\\1&b&c\end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix}1&a&b\\0&c-a&a-b\\0&b-a&c-b\end{vmatrix}=0$$

$[$applying $R_2\to R_2-R_1, R_3\to R_3-R_1]$

$\Rightarrow \quad (c-a)(c-b)+(a-b)^2=0$

$\Rightarrow \quad a^2+b^2+c^2-ab-bc-ca=0$

$\Rightarrow \quad 2a^2+2b^2+2c^2-2ab-2bc-2ca=0$

$\Rightarrow \quad (a-b)^2+(b-c)^2+(c-a)^2=0$

$\Rightarrow \quad a=b=c$

$\Rightarrow ABC$ is an equilateral triangle.

$\Rightarrow A=B=C=\dfrac{\pi}{3}$

$\therefore \sin^2A+\sin^2B+\sin^2C=3\sin^2\dfrac{\pi}{3}=\dfrac{9}{4}$

7. ***(c)*** We have, $MN=NM$

Now, $M^2N^2(M^TN)^{-1}(MN^{-1})^T$

$=M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^TM^T$

$=M^2N(M^T)^{-1}(N^T)^{-1}M^T$

$=M^2N(-M)^{-1}(-N)^{-1}(-M)$

$[\because M^T=-M, N^T=-N]$

$=-M^2NM^{-1}N^{-1}M$

$=-M(MN)(NM)^{-1}M$

$=-M(NM)(NM)^{-1}M$

$=-MM=-M^2$

8. ***(c)*** Let $M=\begin{bmatrix}a&b&c\\x&y&z\\l&m&n\end{bmatrix}$

Then, $M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}\Rightarrow\begin{bmatrix}b\\y\\m\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$

$\Rightarrow \quad b=-1, y=2, m=3$

$M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$

$\Rightarrow \quad a-b=1, x-y=1, l-m=-1$

$\Rightarrow \quad a=0, x=3, l=2$

and $M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}$

$\Rightarrow a+b+c=0, x+y+z=0, l+m+n=12$

9. ***(b)*** The given system of equations will have no solution, if

$$\frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$$

$\Rightarrow \quad \frac{k+1}{k} = \frac{8}{k+3}$ and $\frac{8}{k+3} \neq \frac{4k}{3k-1}$

$\Rightarrow \quad k^2 + 4k + 3 = 8k$ and $24k - 8 \neq 4k^2 + 12k$

$\Rightarrow \quad k^2 - 4k + 3 = 0$ and $4k^2 - 12k + 8 \neq 0$

$\Rightarrow \quad k = 1, 3$ and $k^2 - 3k + 2 \neq 0$

$\Rightarrow \quad k = 3$

Hence, there is only one value of k.

10. ***(d)*** $BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$

$= (A^{-1}A^T)[(A^T)^T(A^{-1})^T]$

$= (A^{-1}A^T)[A(A^{-1})^T]$ $\quad [\because (A^T)^T = A]$

$= A^{-1}(A^TA)(A^{-1})^T$

$= A^{-1}(AA^T)(A^T)^{-1}$

$[\because A^TA = AA^T$, given and $(A^{-1})^T = (A^T)^{-1}]$

$= (A^{-1}A)[A^T(A^T)^{-1}]$

$= I \times I = I$

11. ***(1)*** We have, $\omega = e^{i2\pi/3}$

and $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

$\Rightarrow \quad z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$

$\Rightarrow \quad z[(z+\omega^2)(z+\omega) - 1 - \omega(z+\omega-1) + \omega^2(1-z-\omega^2)] = 0$

$\Rightarrow \quad z^3 = 0$

$\Rightarrow \quad z = 0$ is only one solution.

12. ***(3)*** Let $U_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, U_2 = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ and $U_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Then, $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ 2a+b \\ 3a+2b+c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad a = 1, b = -2, c = 1$

$AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} p \\ 2p+q \\ 3p+2q+r \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$\Rightarrow \quad p = 2, q = -1, r = -4$

and $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ 2x+y \\ 3x+2y+z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$\Rightarrow \quad x = 2, y = -1, z = -3$

$\therefore \quad U = [U_1 \quad U_2 \quad U_3] = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$

$\Rightarrow \quad |U| = \begin{vmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & 0 \\ 1 & -4 & 1 \end{vmatrix}$ $\quad$ [applying $C_3 \to C_3 - C_2$]

$= 3$

13. ***(12)*** A symmetric matrix is symmetric about its diagonal. So, there are even number of 1 and even number of 0 as off diagonal entries. Consequently, there can be either three 1 in the diagonal or one 1 and two zeroes. Thus, we have following two cases

***Case* I** When diagonal elements are 1, 1, 1.

In this case, we have

Number of symmetric matrices

= Number of arrangements 1, 0, 0 or elements above the diagonal

$= \frac{3!}{2!} = 3$

***Case* II** When diagonal element are 1, 0, 0.

In this case, we have

Number of symmetric matrices

= Number of arrangements of 1, 0, 0 in diagonal $\times$ Number of arrangements of 1, 1, 0 or entries above the diagonal

$= \frac{3!}{2!} \times \frac{3!}{2!} = 9$

$\therefore$ Total number of matrices in $A = 3 + 9 = 12$

14. ***(2)*** Given,

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ reduces the determinant into $\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$

or $(p-a)(q-b)r + a(b-q)(c-r) - b(p-a)(c-r) = 0$

Dividing through by $(p-a)(q-b)(r-c)$, we get

$$\frac{r}{r-c} + \frac{a}{p-a} + \frac{b}{q-b} = 0$$

$\Rightarrow \quad \frac{r}{r-c} + \frac{a}{p-a} + 1 + \frac{b}{q-b} + 1 = 2$

$\Rightarrow \quad \frac{r}{r-c} + \frac{p}{p-a} + \frac{q}{q-b} = 2$

15. ***(c, d)***

(a) $(Y^3Z^4 - Z^4Y^3)^T$

$= (Y^3Z^4)^T - (Z^4Y^3)^T$

$= (Z^T)^4(Y^T)^3 - (Y^T)^3(Z^T)^4$

$= -Z^4Y^3 + Y^3Z^4 = Y^3Z^4 - Z^4Y^3$

$\therefore$ It is a symmetric matrix.

(b) $(X^{44} + Y^{44})^T$

$= (X^T)^{44} + (Y^T)^{44}$

$= (-X)^{44} + (-Y)^{44} = X^{44} + Y^{44}$

$\therefore$ It is a symmetric matrix.

$= (-X)^{44} + (-Y)^{44} = X^{44} + Y^{44}$

$\therefore$ It is a symmetric matrix.

(c) $(X^4Z^3 - Z^3X^4)^T = (X^4Z^3)^T - (Z^3X^4)^T$

$= (Z^T)^3(X^T)^4 - (X^T)^4(Z^T)^3 = Z^3X^4 - X^4Z^3$

$= -(X^4Z^3 - Z^3X^4)$

$\therefore$ It is a skew-symmetric matrix.

(d) $(X^{23} + Y^{23})^T = (X^T)^{23} + (Y^T)^{23}$

$= (-X)^{23} + (-Y)^{23} = -(X^{23} + Y^{23})$

$\therefore$ It is a skew-symmetric matrix.

16. ***(b, c)*** We have, $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$

Applying $R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$,

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - R_2$,

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

Applying $C_3 \to C_3 - C_2, C_2 \to C_2 - C_1$,

$$\begin{vmatrix} (1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha) \\ 3+2\alpha & 2\alpha & 2\alpha \\ 2 & 0 & 0 \end{vmatrix} = -648\alpha$$

$\Rightarrow \quad 2\alpha^2(2+3\alpha) - 2\alpha^2(2+5\alpha) = -324\alpha$

$\Rightarrow \quad -4\alpha^3 = -324\alpha$

$\Rightarrow \quad \alpha = 0, \pm 9$

17. ***(d)*** We have,

$a + 8b + 7c = 0,\ 9a + 2b + 3c = 0$

and $a + b + c = 0$

On solving these equations, we get

$b = 6a \Rightarrow c = -7a$

Now, $2x + y + z = 1$

$\Rightarrow \quad 2a + 6a + (-7a) = 1$

$\Rightarrow \quad a = 1, b = 6, c = -7$

$\therefore \quad 7a + b + c = 7(1) + 6 + (-7) = 6$

18. ***(a)*** $a = 2$, b and c satisfies E

Also, $b = 12, c = -14$

$$\therefore \frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}}$$

$$= 3\omega + 1 + 3\omega^2 = -2$$

19. ***(b)*** Given, $ax^2 + bx + c = 0$

$\Rightarrow \quad x^2 + 6x - 7 = 0$

$\Rightarrow \quad \alpha = 1, \beta = -7$

$$\therefore \quad \sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{1} - \frac{1}{7}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n$$

$$= \frac{1}{1 - \frac{6}{7}} = 7$$

20. $A \to r; B \to q; C \to p; D \to s$

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= -\frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$$

A. If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$, then $\Delta = 0$ and $a = b = c \neq 0$

Therefore, the equations represent identical planes.

B. $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca \Rightarrow \Delta = 0$

Therefore, the equations have infinitely many solutions.

$\therefore \quad ax + by = (a+b)z$

$\Rightarrow \quad bx + cy = (b+c)z$

$\Rightarrow \quad (b^2 - ac)y = (b^2 - ac)z$

$\Rightarrow \quad y = z$

$\Rightarrow \quad ax + by + cy = 0$

$\Rightarrow \quad ax = ay \Rightarrow x = y = z$

C. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca \Rightarrow \Delta = 0$

Notice Board

JEE 2017 WITH NEW GUIDLINES

The Indian Institute of Technology_ Madras will be conducting the Joint Entrance Examination (JEE) Advanced 2017. The dates for the main examination is not released by the CBSE yet.

The JEE cell at IIT Madras is in the process of finalising the events required for conducting the examination. Last year, IIT Guwahati conducted the JEE.

The Union Ministry of Human Resource Development (HRD Ministry) has decided that class 12th marks will not be a factor for determining rankings in JEE for admissions to engineering undergraduate courses from 2017. The present system of allotting 40% weightage to class 12th marks for determining ranks in JEE shall be dispensed.

150 Integer Ques, **400** Passages, **450** More Than one Option, **200** Matrix Matching in **PHYSICS**

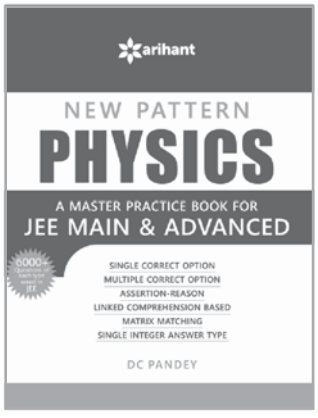
850 Integer Ques, **400** Passages, **850** More Than one Option, **200** Matrix Matching in **CHEMISTRY**

400 Integer Ques, **650** Passages, **650** More Than one Option, **150** Matrix Matching in **MATHEMATICS**

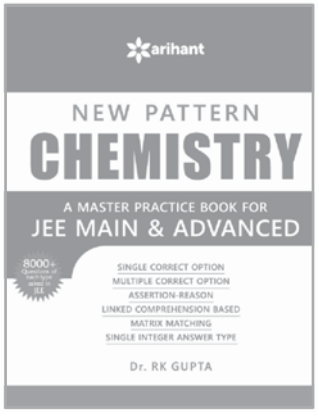
arihant

NEW PATTERN BOOKS

FOR JEE MAIN & ADVANCED



B062 ₹665.00



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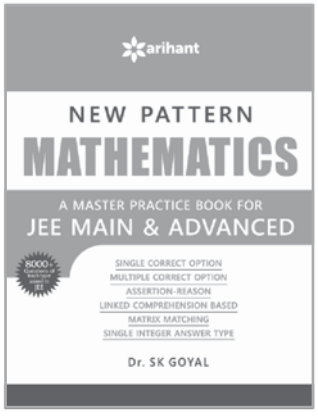

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EACH BOOK HAVING THOUSAND OF JEE LEVEL QUESTIONS OF ALL FORMATS

- SINGLE CORRECT OPTION
- MULTIPLE CORRECT OPTIONS
- ASSERTION-REASON
- LINKED COMPREHENSION BASED
- MATRIX MATCHING
- SINGLE INTEGER ANSWER TYPE

Love Agarwal

DIFFERENTIAL EQUATION

Coverage of special topics for JEE Advanced

In this article, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation, some methods to solve a first order-first degree differential equation and some applications of differential equations in different areas.

Differential Equation of First Order and First Degree

An equation that involves independent and dependent variables and the derivatives of the dependent variables with respect to independent variable is called a differential equation.

e.g. $\frac{dy}{dx} = x^2 \log x, dy = \sin x\, dx$, $y = x^2 \frac{dy}{dx} + 9$

Order and Degree of Differential Equation

The ***order*** of a differential equation is the order of the highest differential coefficient occurring in it.

The ***degree*** of a differential equation which is expressed or can be expressed as a polynomial in the derivatives is the degree of the highest order derivative occurring in it after it has been expressed in a form free from radicals and fractions as far as derivatives are concerned.

Example 1 The order and degree of the differential equation $\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{5/3}$ are

[IIT JEE 2004]

(a) 3, 2 (b) 2, 3 (c) 1, 2 (d) 2, 1

Sol. **(b)** We have,

$$\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{5/3}$$

$$\therefore \quad \left(\frac{d^2y}{dx^2}\right)^3 = \left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^5$$

Hence, order is 2 and degree is 3.

Formation of Differential Equations

Consider a family of curves

$$f(x, y, \alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \qquad \ldots(i)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n independent parameters. Eq. (i) is known as an n parameter family of curves. If we differentiate Eq. (i) n times w.r.t. x, we will get n more relations between $x, y, \alpha_1, \alpha_2, \ldots, \alpha_n$ and derivatives of y with respect to x. By eliminating $\alpha_1, \alpha_2, \ldots, \alpha_n$ from these n relations and Eq. (i), we get a differential equation.

Clearly, the order of this differential equation will be n, i.e. equal to the number of independent parameters in the family of curves.

Example 2 The order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x + c_5}$, where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants, is

[IIT JEE 1998]

(a) 5 (b) 4 (c) 3 (d) 2

Sol. **(c)** Given, $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x + c_5}$ ...(i)

$\Rightarrow \quad y = (c_1 + c_2)\cos(x + c_3) - c_4 e^x \cdot e^{c_5}$

Now, let $c_1 + c_2 = A, c_3 = B, c_4 e^{c_5} = c$

$\Rightarrow \quad y = A\cos(x + B) - ce^x$ ...(ii)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin(x + B) - ce^x \qquad \ldots(iii)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -A\cos(x + B) - ce^x \qquad \ldots(iv)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y - 2ce^x \qquad \ldots(v)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = -2ce^x$$

Again, differentiating w.r.t. x, we get

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} = -2ce^x \qquad \ldots(vi)$$

$$\Rightarrow \quad \frac{d^3y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dx^2} + y \qquad [\text{from Eq. (v)}]$$

which is a differential equation of order 3.

Alternate Solution

Given, $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x + c_5}$

$\Rightarrow \quad y = A\cos(x + B) - ce^x$

where, $c_1 + c_2 = A, c_3 = B, c_4 e^{c_5} = c$

Since, given equation has 3 parameters, so its order will be 3.

Example 3 The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, is of [IIT JEE 1999]

(a) order 1 (b) order 2
(c) degree 3 (d) degree 4

Sol. ***(a, c)*** Given, $y^2 = 2c\ (x + \sqrt{c})$ ...(i)

On differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 2c \Rightarrow c = y\frac{dy}{dx}$$

On putting this value of c in Eq. (i), we get

$$y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{y\frac{dy}{dx}}\right)$$

$$\Rightarrow \quad y = 2\frac{dy}{dx}\cdot x + 2y^{1/2}\left(\frac{dy}{dx}\right)^{3/2}$$

$$\Rightarrow \quad y - 2x\frac{dy}{dx} = 2\sqrt{y}\left(\frac{dy}{dx}\right)^{3/2}$$

$$\Rightarrow \quad \left(y - 2x\frac{dy}{dx}\right)^2 = 4y\left(\frac{dy}{dx}\right)^3$$

Therefore, the order of this differential equation is 1 and degree is 3.

Example 4 Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' $\left(\text{here, } y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right)$, then which of the following statement(s) is(are) true? [JEE Advanced 2015]

(a) $P = y + x$
(b) $P = y - x$
(c) $P + Q = 1 - x + y + y' + (y')^2$
(d) $P - Q = x + y - y' - (y')^2$

Sol. ***(b, c)*** Centres of circle lie on the straight line $y = x$.

$\because$ Equation of family of circles is

$$(x - \alpha)^2 + (y - \alpha)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2\alpha x - 2\alpha y + 2\alpha^2 - r^2 = 0$$

On differentiating w.r.t. x, we get

$$2x + 2yy' - 2\alpha - 2\alpha y' = 0$$

$$\Rightarrow \quad \alpha = \frac{x + yy'}{1 + y'}$$

Again, differentiating w.r.t. x, we get

$$2 + 2(y')^2 + 2yy'' - 2\alpha y'' = 0$$

$$\Rightarrow \quad 1 + (y')^2 + yy'' - \left(\frac{x + yy'}{1 + y'}\right)\cdot y'' = 0$$

$$\Rightarrow 1 + y' + (y')^2 + (y')^3 + yy'' + yy'y'' - xy'' - yy'y'' = 0$$

$$\Rightarrow (y - x)y'' + (1 + y' + (y')^2)y' + 1 = 0$$

$$\Rightarrow \quad P = y - x, Q = 1 + y' + (y')^2$$

$$\therefore \quad P + Q = 1 - x + y + y' + (y')^2$$

Solution of a Differential Equation

A solution of a differential equation is an equation which contains arbitrary constants as many as the order of the differential equation and is called the general solution. Other solutions, obtained by giving particular values to the arbitrary constants in the general solution, are called particular solutions.

Also, we know that the general integral of a function contains an arbitrary constant. Therefore, the solution of a differential equation, resulting as it does from the operations of integration, must contain arbitrary constants, equal in number to the number of times the integration is involved in obtaining the solution and this latter is equal to the order of the differential equation.

Thus, we can conclude that the general solution of a differential equation of the nth order must contain n and only n independent arbitrary constants.

Method of Variables Separation

If the coefficient of dx is only a function of x and dy is only a function of y in the given differential equation, then the equation can be solved using variable separation method. Thus, the general form of such an equation is

$$f(x)\,dx + g(y)\,dy = 0 \qquad \ldots(i)$$

On integrating, we get $\int f(x)\,dx + \int g(y)\,dy = c$

Which is a general solution of Eq. (i).

Example 5 The differential equation $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$ determines a family of circles with [IIT JEE 2000]

(a) variable radii and a fixed centre at (0, 1)
(b) variable radii and a fixed centre at (0, –1)
(c) fixed radius 1 and variable centres along the X-axis
(d) fixed radius 1 and variable centres along the Y-axis

Sol. **(c)** Given, $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y} \Rightarrow \int \frac{y}{\sqrt{1-y^2}}\,dy=\int dx$

$\Rightarrow \quad -\sqrt{1-y^2}=x+C$

$\Rightarrow \quad (x+C)^2+y^2=1$

Hence, the above equation represents a circle with centre at $(-c, 0)$ on X-axis and radius = 1.

Example 6 If $y=y(x)$ and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right)=-\cos x$, $y(0)=1$, then $y\left(\frac{\pi}{2}\right)$ equals to [IIT JEE 2004]

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{1}{3}$ (d) 1

Sol. **(a)** Given, $\frac{dy}{dx}=\frac{-\cos x\,(y+1)}{2+\sin x} \Rightarrow \frac{dy}{y+1}=\frac{-\cos x}{2+\sin x}\,dx$

On integrating both sides, we get

$$\int \frac{dy}{y+1}=-\int \frac{\cos x}{2+\sin x}\,dx$$

$\Rightarrow \quad \log(y+1)=-\log(2+\sin x)+\log C$

When, $x=0,\ y=1$, then $C=4$

$\Rightarrow \quad y+1=\frac{4}{2+\sin x}$

$\therefore \quad y\left(\frac{\pi}{2}\right)=\frac{4}{3}-1=\frac{1}{3}$

Example 7 For the following question

Let a solution $y=y(x)$ of the differential equation $x\sqrt{x^2-1}\,dy-y\sqrt{y^2-1}\,dx=0$ satisfy $y(2)=\frac{2}{\sqrt{3}}$.

Statement I $y(x)=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)$

Statement II $y(x)$ is given by $\frac{1}{y}=\frac{2\sqrt{3}}{x}-\sqrt{1-\frac{1}{x^2}}$.

Choose the correct answer from the codes (a), (b), (c) and (d) defined as follows.

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true [IIT JEE 2007]

Sol. **(c)** Given, $\frac{dy}{dx}=\frac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}}$

$\Rightarrow \quad \int \frac{dy}{y\sqrt{y^2-1}}=\int \frac{dx}{x\sqrt{x^2-1}}$

$\Rightarrow \quad \sec^{-1}y=\sec^{-1}x+C$

At $x=2,\ y=\frac{2}{\sqrt{3}};\ \frac{\pi}{6}=\frac{\pi}{3}+C \Rightarrow C=-\frac{\pi}{6}$

Now, $y=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)=\cos\left[\cos^{-1}\frac{1}{x}-\cos^{-1}\frac{\sqrt{3}}{2}\right]$

$$=\cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x}+\sqrt{1-\frac{1}{x^2}}\sqrt{1-\frac{3}{4}}\right)\right]$$

$$=\frac{\sqrt{3}}{2x}+\frac{1}{2}\sqrt{1-\frac{1}{x^2}}$$

Homogeneous Equations

The function $f(x, y)$ is said to be a homogeneous function of degree n, if for any real number $t\,(\neq 0)$, we have

$$f(tx, ty)=t^n f(x, y)$$

A differential equation of the form $\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$, where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x and y, and of the same degree, is called homogeneous. This equation may also be reduced to the form $\frac{dy}{dx}\Rightarrow g\left(\frac{y}{x}\right)$ and is solved by putting $y=vx$, so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variable separable.

Example 8 A curve passes through the point $(1, \pi/6)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right)$, $x>0$. Then, the equation of the curve is [JEE Advanced 2013]

(a) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$

(c) $\sec\left(\frac{2y}{x}\right)=\log x+2$ (d) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

Sol. **(a)** It is given that,

$$\frac{dy}{dx}=\frac{y}{x}+\sec\left(\frac{y}{x}\right),\ x>0 \qquad \ldots(i)$$

Let $y=vx$. Then, $\frac{dy}{dx}=v+x\frac{dv}{dx}$

On substituting these values in Eq. (i), we obtain

$$v+x\frac{dv}{dx}=v+\sec v$$

$\Rightarrow \quad x\frac{dv}{dx}=\sec v$

$\Rightarrow \quad \cos v\,dv=\frac{1}{x}\,dx$

On integrating both sides, we get

$$\sin v = \log x + C$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = \log x + C \qquad \ldots(ii)$$

It passes through the point $\left(1, \frac{\pi}{6}\right)$.

$$\therefore \quad \sin\frac{\pi}{6} = \log 1 + C \Rightarrow C = \frac{1}{2}$$

Putting $C = \frac{1}{2}$ in Eq. (ii), we obtain the equation of the curve as $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$.

Example 9 For the primitive integral equation $(x^2 + y^2)\,dy = xy\,dx$, the solution is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$, then x_0 is [IIT JEE 2005]

(a) $\sqrt{2(e^2 - 1)}$ (b) $\sqrt{2(e^2 + 1)}$

(c) $\sqrt{3}e$ (d) $\sqrt{\frac{e^2 + 1}{2}}$

Sol. ***(c)*** The given differential equation is

$$(x^2 + y^2)dy = xy\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy}{x^2 + y^2} \qquad \ldots(i)$$

which is an homogeneous equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2 v}{x^2 + x^2 v^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1 + v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^3}{1 + v^2}$$

$$\Rightarrow \quad \int \frac{1 + v^2}{v^3}\,dv + \int \frac{dx}{x} = 0$$

$$\Rightarrow \quad \frac{-1}{2v^2} + \log v + \log x = C$$

$$\Rightarrow \quad \frac{-x^2}{2y^2} + \log y = C$$

$$\Rightarrow \quad \log y = C + \frac{x^2}{2y^2}$$

Also, $y(1) = 1$

$$\therefore \quad \log 1 = C + \frac{1}{2} \Rightarrow C = -\frac{1}{2}$$

$$\therefore \quad \log y = \frac{x^2 - y^2}{2y^2}$$

Given, $y(x_0) = e$

$$\Rightarrow \quad \log e = \frac{x_0^2 - e^2}{2e^2}$$

$$\Rightarrow \quad x_0^2 = 3e^2$$

$$\Rightarrow \quad x_0 = \sqrt{3}e$$

Linear Differential Equation

Equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone, is called linear differential equation.

For solving such equation, we multiply both sides by Integrating Factor $= (\text{IF}) = e^{\int P\,dx}$.

Multiplying given equations by IF, we get

$$e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int P\,dx}$$

$$\Rightarrow \quad \frac{dy}{dx}e^{\int P\,dx} + y\,Pe^{\int P\,dx} = Qe^{\int P\,dx}$$

$$\Rightarrow \quad \frac{d}{dx}(ye^{\int P\,dx}) = Qe^{\int P\,dx}$$

$$\Rightarrow \quad y\,e^{\int P\,dx} = \int Qe^{\int P\,dx} + C$$

which is the required solution of the given differential equation.

In some cases a linear differential equation may be of the form $\frac{dx}{dy} + P_1 x = Q$, where P_1 and Q are functions of y alone or constants.

In such a case the integrating factor is $e^{\int P\,dy}$ and solution is given by

$$x\,e^{\int P\,dx} = \int Q\,e^{\int P\,dy}\,dy + C.$$

Example 10 Let $y(x)$ be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x, x \geq 1$. Then, $y(e)$ is equal to [JEE Main 2015]

(a) e (b) 0 (c) 2 (d) $2e$

Sol. ***(c)*** Given differential equation is

$$(x\log x)\frac{dy}{dx} + y = 2x\log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x\log x} = 2$$

Which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Now, the solution of given differential equation is given by

$$y\log x = \int (\log x)(2\,dx)$$

$$\Rightarrow \quad y\log x = 2\int \log x\,dx$$

$$\Rightarrow \quad y\log x = 2(x\log x - x) + C$$

It is given that $x \geq 1$. Putting $x = 1$,

we obtain $C = 2$

$$\therefore \quad y\log x = 2(x\log x - x) + 2$$

At $x = e$, $y = 2(e - e) + 2 \Rightarrow y = 2$

Example 11 The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1,1)$ satisfying $f(0)=0$. Then, $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)dx$ is [JEE Advanced 2014]

(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Sol. ***(b)*** Given differential equation is $\frac{dy}{dx} + \frac{x}{(x^2-1)}y = \frac{x^4+2x}{\sqrt{1-x^2}}$.

Which is a linear differential equation.

$\therefore$ IF $= e^{\int \frac{x}{x^2-1}dx} = e^{\frac{1}{2}\log|x^2-1|} = \sqrt{1-x^2}$ $[\because x \in (-1, 1)]$

So, the required solution is

$$y\sqrt{1-x^2} = \int \frac{x^4+2x}{\sqrt{1-x^2}}\sqrt{1-x^2}\,dx + C$$

$$\Rightarrow \quad y\sqrt{1-x^2} = \int (x^4+2x)\,dx + C$$

$$\Rightarrow \quad y\sqrt{1-x^2} = \frac{x^5}{5} + x^2 + C$$

It is given that, $f(0) = 0$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad f(x)\sqrt{1-x^2} = \frac{x^5}{5} + x^2$$

Now, $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{\left(\frac{x^5}{5}+x^2\right)}{\sqrt{1-x^2}}dx$

$$= \frac{1}{5}\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^5}{\sqrt{1-x^2}}dx + \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx$$

$$= 2\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx$$

Putting $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$\therefore \quad \int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)dx = 2\int_0^{\pi/3} \sin^2\theta\, d\theta$$

$$= \int_0^{\pi/3} (1-\cos 2\theta)\,d\theta = \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$= \frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Example 12 Let $f:\left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then, the value of $\int_{1/2}^{1} f(x)\,dx$ lies in the interval [JEE Advanced 2013]

(a) $(2e-1, 2e)$ (b) $(e-1, 2e-1)$

(c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

Sol. ***(d)*** Here, $f'(x) < 2f(x)$

$$\Rightarrow \quad f'(x) - 2f(x) < 0$$

$$\text{IF} = e^{\int -2dx} = e^{-2x}$$

On multiplying by e^{-2x}, we get

$$f'(x)e^{-2x} - 2e^{-2x}f(x) < 0$$

$$\Rightarrow \quad \frac{d}{dx}\{f(x)\cdot e^{-2x}\} < 0$$

$\therefore$ $\phi(x) = f(x)e^{-2x}$ is decreasing for $x \in \left[\frac{1}{2}, 1\right]$.

When $x > \frac{1}{2}$, then

$$\phi(x) < \phi\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad e^{-2x}f(x) < e^{-1}f\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad f(x) < e^{2x-1}\cdot 1 \qquad \left[\because f\left(\frac{1}{2}\right) = 1\right]$$

$$\Rightarrow \quad 0 < \int_{1/2}^{1} f(x)\,dx < \int_{1/2}^{1} e^{2x-1}dx$$

$$\Rightarrow \quad 0 < \int_{1/2}^{1} f(x)\,dx < \left(\frac{e^{2x-1}}{2}\right)_{1/2}^{1}$$

$$\Rightarrow \quad 0 < \int_{1/2}^{1} f(x)\,dx < \frac{e^1 - e^0}{2}$$

$$\Rightarrow \quad 0 < \int_{1/2}^{1} f(x)\,dx < \frac{e-1}{2}$$

Example 13 If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to [IIT JEE 2003]

(a) $-\frac{1}{2}$ (b) $e + \frac{1}{2}$

(c) $e - \frac{1}{2}$ (d) $\frac{1}{2}$

Sol. ***(a)*** Given, $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$

which is a linear differential equation.

$$\therefore \quad \text{IF} = e^{\int \frac{-t}{1+t}dt} = e^{-t}(1+t)$$

$\therefore$ Required solution is

$$ye^{-t}(1+t) = \int \frac{1}{1+t}\cdot e^{-t}(1+t)\,dt + C$$

$$\Rightarrow \quad ye^{-t}(1+t) = -e^{-t} + C$$

At $t = 0, y = -1$

$$\Rightarrow \quad -1e^0(1+0) = -e^0 + C$$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad y = \frac{-1}{1+t}$$

Now, $y(1) = \frac{-1}{2}$

Example 14 If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then [IIT JEE 2012]

(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

Sol. ***(a, d)*** We have,

$$y' - y\tan x = 2x\sec x \text{ and } y(0) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} - y\tan x = 2x\sec x$$

$$\therefore \quad \text{IF} = \int e^{-\tan x}\,dx = e^{\log|\cos x|} = \cos x$$

$$\therefore \text{Solution is } y\cdot\cos x = \int 2x\sec x\cdot\cos x\,dx + C$$

$$\Rightarrow \quad y\cos x = x^2 + C$$

As $\quad y(0) = 0 \Rightarrow C = 0$

$$\therefore \quad y = x^2\sec x$$

Now, $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}\sec\frac{\pi}{4} = \frac{\pi^2}{16}\times\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$

Again, $y'(x) = 2x\sec x + x^2\sec x\tan x$

$$\Rightarrow \quad y'\left(\frac{\pi}{4}\right) = 2\times\frac{\pi}{4}\times\sqrt{2} + \frac{\pi^2}{16}\times\sqrt{2}\times 1 = \frac{\pi}{\sqrt{2}} + \frac{\pi^2}{8\sqrt{2}}$$

Also, $\quad y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9}$ and $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

Example 15 Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then, the value of $y(2)$ is ..... . [IIT JEE 2011]

Sol. ***(0)*** We have,

$$\frac{dy}{dx} + yg'(x) = g(x)g'(x)$$

$$\therefore \quad \text{IF} = e^{\int g'(x)dx} = e^{g(x)}$$

$$\therefore \text{Solution is } ye^{g(x)} = \int g(x)g'(x)e^{g(x)}dx + C \qquad \ldots(i)$$

Put $\quad g(x) = t \Rightarrow g'(x)\,dx = dt$

From Eq. (i),

$$ye^{g(x)} = \int t\cdot e^t dt + C = te^t - e^t + C$$

$$\Rightarrow \quad ye^{g(x)} = \{g(x) - 1\}e^{g(x)} + C$$

Given, $\quad y(0) = 0,\ g(0) = g(2) = 0$

$\therefore$ Eq. (i) becomes

$$y(0)e^{g(0)} = (g(0) - 1)e^{g(0)} + C$$

$$\Rightarrow \quad 0 = (-1)\cdot 1 + C \Rightarrow C = 1$$

$$\therefore \quad y(x)\cdot e^{g(x)} = \{g(x) - 1\}e^{g(x)} + 1 \qquad \ldots(ii)$$

On putting $x = 2$ in Eq. (ii), we get

$$y(2)e^{g(2)} = \{g(2) - 1\}e^{g(2)} + 1$$

$$\Rightarrow \quad y(2)e^0 = (0 - 1)e^0 + 1 \qquad [\because g(2) = 0]$$

$$\Rightarrow \quad y(2) = (-1) + 1 = 0$$

Applications of Differential Equations

Following are some applications, where differential equations are used

1. Growth and Decay Problems
2. Temperature problems
3. Dilution Problems
4. Orthogonal Trajectory
5. Geometrical Application

Example 16 Let the population of rabbits surviving at a time t be governed by the differential equations $\frac{dP}{dt} = \frac{1}{2}P(t) - 200$. If $P(0) = 100$, then $P(t)$ is equal to [JEE Main 2004]

(a) $400 - 300e^{t/2}$ (b) $300 - 200e^{-t/2}$

(c) $600 - 500e^{t/2}$ (d) $400 - 300e^{-t/2}$

Sol. ***(a)*** Given differential equation is $\frac{dP}{dt} - \frac{1}{2}P(t) = -200$ is a linear differential equation.

Here, $\quad P = -\frac{1}{2},\ Q = -200$

$$\therefore \quad \text{IF} = e^{\int -\frac{1}{2}dt} = e^{-\frac{t}{2}}$$

Hence, the required solution is

$$P(t)\cdot\text{IF} = \int Q\cdot\text{IF}\,dt + C$$

$$\Rightarrow \quad P(t)\cdot e^{-\frac{t}{2}} = \int -200\,e^{-\frac{t}{2}}dt + C$$

$$\Rightarrow \quad P(t)e^{-\frac{t}{2}} = 400\,e^{-\frac{t}{2}} + C \Rightarrow P(t) = 400 + Ce^{\frac{t}{2}}$$

Since, $\quad P(0) = 100 \Rightarrow C = -300$

$$\therefore \quad P(t) = 400 - 300e^{\frac{t}{2}}$$

Example 17 At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employees 25 more workers, then the new level of production of items is [JEE Main 2013]

(a) 2500 (b) 3000

(c) 3500 (d) 4500

Sol. ***(c)*** Given, $\frac{dP}{dx} = 100 - 12\sqrt{x}$

$$\Rightarrow \quad dP = (100 - 12\sqrt{x})\,dx$$

On integrating both the sides, we get

$$\int dP = \int (100 - 12\sqrt{x})dx$$

$$\Rightarrow \quad P = 100x - 8x^{3/2} + C$$

When $\quad x = 0$, then $P = 2000$

$$\Rightarrow \quad C = 2000$$

Now, when $x = 25$, then

$$P = 100 \times 25 - 8\,(25)^{3/2} + 2000$$
$$= 2500 - 8 \times 125 + 2000 = 3500$$

Example 18 Tangent is drawn at any point P of a curve which passes through (1, 1) cutting X-axis and Y-axis at A and B, respectively. If $BP : AP = 3 : 1$, then

(a) differential equation of the curve is $3x\dfrac{dy}{dx} + y = 0$

(b) differential equation of the curve is $3x\dfrac{dy}{dx} - y = 0$

(c) curve is passing through $\left(\dfrac{1}{8}, 2\right)$

(d) normal at (1, 1) is $x + 3y = 4$

Sol. ***(a, c)*** Since, $BP : AP = 3 : 1$, then equation of tangent is

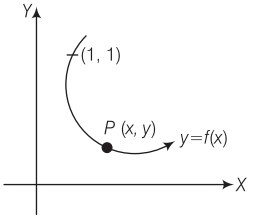
$$Y - y = f'(x)(X - x)$$

The intercept on the coordinate axes are

$$A\left(x - \frac{y}{f'(x)}, 0\right) \text{ and } B(0, y - x f'(x)).$$

Since, P is internally intercepts a line AB, then

$$x = \frac{3\left(x - \dfrac{y}{f'(x)}\right) + 1 \times 0}{3 + 1}$$



$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{-3x} \Rightarrow \frac{dy}{y} = -\frac{1}{3x}dx$$

On integrating both the sides, we get

$$xy^3 = C$$

Since, the curve passes through (1, 1), then $c = 1$

$$\therefore \quad xy^3 = 1$$

At $\quad x = \dfrac{1}{8}, \; y = 2$

Smart Practice

Single Option Correct Type

1. If m and n are the order and degree of the differential equation $\left(\dfrac{d^2y}{dx^2}\right)^5 + \dfrac{4\left(\dfrac{d^2y}{dx^2}\right)^3}{\dfrac{d^3y}{dx^3}} + \dfrac{d^3y}{dx^3} = x^2 - 1$, then

(a) $m = 3, n = 3$ *(b)* $m = 3, n = 2$

(c) $m = 3, n = 5$ *(d)* $m = 3, n = 1$

2. The degree of the differential equation

$$x = 1 + \frac{dy}{dx} + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \ldots \text{ is}$$

(a) 3 *(b)* 1 *(c)* not defined *(d)* None of these

3. Let F be the family of ellipse, whose centre is the origin and major axis is the Y-axis. Then, the differential equation of family F is

(a) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \left(x\dfrac{dy}{dx} - y\right) = 0$

(b) $xy\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}\left(x\dfrac{dy}{dx} - y\right) = 0$

(c) $xy\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\left(x\dfrac{dy}{dx} - y\right) = 0$

(d) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}\left(x\dfrac{dy}{dx} - y\right) = 0$

4. The orthogonal trajectories of the family of curves $a^{n-1}y = x^n$ are given by (a is the arbitrary constant)

(a) $x^n + n^2y = \text{Constant}$

(b) $ny^2 + x^2 = \text{Constant}$

(c) $n^2x + y^n = \text{Constant}$

(d) $n^2x - y^n = \text{Constant}$

5. The solution of differential equation

$$yy' = x\left(\frac{y^2}{x^2} + \frac{f(y^2/x^2)}{f'(y^2/x^2)}\right) \text{ is}$$

(a) $f\left(\dfrac{y^2}{x^2}\right) = Cx^2$ *(b)* $x^2 f\left(\dfrac{y^2}{x^2}\right) = C^2y^2$

(c) $x^2 f\left(\dfrac{y^2}{x^2}\right) = C$ *(d)* $f\left(\dfrac{y^2}{x^2}\right) = \dfrac{Cy}{x}$

6. The solution of the differential equation $x^2\dfrac{dy}{dx}\cos\left(\dfrac{1}{x}\right) - y\sin\left(\dfrac{1}{x}\right) = -1$, where $y \to -1$ as $x \to \infty$ is

(a) $y = \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right)$ *(b)* $y = \dfrac{x+1}{x\sin\left(\dfrac{1}{x}\right)}$

(c) $y = \cos\left(\dfrac{1}{x}\right) + \sin\left(\dfrac{1}{x}\right)$ *(d)* $y = \dfrac{x+1}{x\cos\left(\dfrac{1}{x}\right)}$

7. The solution of differential equation

$$x^2 = 1 + \left(\frac{x}{y}\right)^{-1}\frac{dy}{dx} + \frac{\left(\frac{x}{y}\right)^{-2}\left(\frac{dy}{dx}\right)^2}{2!} + \frac{\left(\frac{x}{y}\right)^{-3}\left(\frac{dy}{dx}\right)^3}{3!} + \ldots$$

is

(a) $y^2 = x^2(\ln x - 1) + C$ (b) $y = x^2(\ln x^2 - 1) + C$
(c) $y^2 = x(\ln x - 1) + C$ (d) $y = x^2 e^{x^2} + C$

8. An object falling from rest in air is subject not only to the gravitational force but also to air resistance. Assume that the air resistance is proportional to the velocity with constant of proportionality as $k > 0$ and acts in a direction opposite to motion [$g = 9.8$ m/s^2]. Then, velocity cannot exceed

(a) $\left(\frac{9.8}{k}\right)$ m/s (b) $\left(\frac{98}{k}\right)$ m/s
(c) $\left(\frac{k}{9.8}\right)$ m/s (d) None of these

9. The solution of the differential equation $(e^{x^2} + e^{y^2})y\frac{dy}{dx} + e^{x^2}(xy^2 - x) = 0$ is

(a) $e^{x^2}(y^2 - 1) + e^{y^2} = C$ (b) $e^{y^2}(x^2 - 1) + e^{x^2} = C$
(c) $e^{y^2}(y^2 - 1) + e^{x^2} = C$ (d) $e^{x^2}(y - 1) + e^{y^2} = C$

10. The solution of the differential equation $\frac{dy}{dx} = \frac{1}{xy[x^2 \sin y^2 + 1]}$ is

(a) $x^2(\cos y^2 - \sin y^2 - 2Ce^{-y^2}) = 2$
(b) $y^2(\cos x^2 - \sin y^2 - 2Ce^{-y^2}) = 2$
(c) $x^2(\cos y^2 - \sin y^2 - e^{-y^2}) = 4C$
(d) None of the above

11. The solution of $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$ satisfying $y(1) = 1$ is given by

(a) a system of parabolas (b) a system of circles
(c) a system of hyperbola (d) a system of ellipse

12. A normal at $P(x, y)$ on a curve meets the X-axis at Q and N is the foot of the ordinate at P. If $NQ = \frac{x(1 + y^2)}{1 + x^2}$, then the equation of curve, given that it passes through the point (3, 1), is

(a) $x^2 - y^2 = 8$ (b) $x^2 + 2y^2 = 11$
(c) $x^2 - 5y^2 = 4$ (d) None of these

One or More than One Option Correct Type

13. For the differential equation, whose solution is $(x - h)^2 + (y - k)^2 = a^2$ (a is a constant), its

(a) order is 2 (b) order is 3
(c) degree is 2 (d) degree is 3

14. Identify the statement(s) which is/are true.

(a) $f(x, y) = e^{y/x} + \tan\left(\frac{y}{x}\right)$ is a homogeneous of degree zero
(b) $x\ln\left(\frac{y}{x}\right)dx + \frac{y^2}{x}\sin^{-1}\left(\frac{y}{x}\right)dy = 0$ is a homogeneous differential equation
(c) $f(x, y) = x^2 + \sin x \cos y$ is a non-homogeneous
(d) $(x^2 + y^2)dx - (xy^2 - y^3)dy = 0$ is a homogeneous differential equation

15. The solution of the differential equation $(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$ is

(a) $1 + x^2y^2 = Cx$ (b) $1 + x^2y^2 = Cy$
(c) $\frac{1}{y} + x^2y = C$ (d) $y = \frac{-1}{x^2}$

16. The solution of $\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$ is

(a) $\sqrt{x^2 + y^2} = \sin\left\{\tan^{-1}\left(\frac{y}{x}\right) + C\right\}$
(b) $\sqrt{x^2 + y^2} = \cos\left\{\tan^{-1}\left(\frac{y}{x}\right) + C\right\}$
(c) $\sqrt{x^2 + y^2} = \tan\left\{\sin^{-1}\left(\frac{y}{x}\right) + C\right\}$
(d) $y = x\tan\{C + \sin^{-1}\sqrt{x^2 + y^2}\}$

17. In which of the following differential equation degree is not defined?

(a) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x\log\left(\frac{d^2y}{dx^2}\right)$
(b) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$
(c) $x = \sin\left(\frac{dy}{dx} - 2y\right), |x| < 1$
(d) $x - 2y = \log\left(\frac{dy}{dx}\right)$

Linked Comprehension Type

Directions (Q. Nos. 18-20) The differential equation

$$y = p(x) + f(p) \quad \ldots(i)$$

where $p = \frac{dy}{dx}$, is known as Claisalt's equation. To solve Eq. (i), differentiate it with respect to x, which gives either

$$\frac{dp}{dx} = 0 \Rightarrow p = c \quad \ldots(ii)$$

or $$x + f'(p) = 0 \quad \ldots(iii)$$

Note

- If p is eliminated between Eqs. (i) and (ii), then solution obtained is a general solution of Eq. (i).
- If p is eliminated between Eqs. (i) and (iii), then solution obtained does not contain any arbitrary constant and is not particular solution of Eq. (i). This solution is called singular solution of Eq. (i).

18. Which of the following is true about solutions of differential equation $y = xy' + \sqrt{1 + g^2}$?

(a) The general solution of equation is family of parabola
(b) The general solution of equation is family of circles
(c) The singular solution of equation is circle
(d) The singular solution of equation is ellipse

19. The number of solutions of the equation $f(x) = -1$ and the singular solution of the equation $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$ is

(a) 1 *(b)* 2
(c) 4 *(d)* 0

20. The singular solution of the differential equation $y = mx + m - m^3$, where $m = \frac{dy}{dx}$, passes through the point

(a) (0, 0) *(b)* (0, 1)
(c) (1, 0) *(d)* (– 1, 0)

Directions (Q. Nos. 21-23) For certain curves $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ has local minimum value 5, when $x = 1$.

21. The number of critical points for $y = f(x)$ for $x \in [0, 2]$ is

(a) 0 *(b)* 1
(c) 2 *(d)* 3

22. Global minimum value of $y = f(x)$ for $x \in [0, 2]$ is

(a) 5 *(b)* 7
(c) 8 *(d)* 9

23. Global maximum value of $y = f(x)$ for $x \in [0, 2]$ is

(a) 5 *(b)* 7 *(c)* 8 *(d)* 9

Match the Columns

24. Match the following Columns.

	Column I		Column II (Differential equation)
A.	Order 1	p.	of all parabolas, whose axis is the X-axis
B.	Order 2	q.	of family of curves $y = a(x + a)^2$, where a is an arbitrary constant
C.	Degree 1	r.	$\left(1 + 3\frac{dy}{dx}\right)^{2/3} = 4\frac{d^3y}{dx^3}$
D.	Degree 3	s.	of family of curve $y^2 = 2c(x + \sqrt{c})$, where $c > 0$

25. Match the following Columns.

	Column I		Column II
A.	If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\frac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = k$, then the value of $\frac{k}{3}$ is	p.	3
B.	Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is	q.	4
C.	If real value of m for which the substitution $y = u^m$ will transform the differential equation $2x^4 y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation, then the value of $2m$ is	r.	2
D.	If the solution of differential equation $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 12y$ is $y = Ax^m + Bx^{-n}$, then $\lvert m + n \rvert$ is	s.	1

Single Integer Type

26. If $x\frac{dy}{dx} = x^2 + y - 2$, $y(1) = 1$, then $y(2)$ equals to .....

27. Let $y = y(t)$ be a solution to the differential equation $y' + 2ty = t^2$, then $16\lim_{t\to\infty}\frac{y}{t}$ is equal to .....

28. If the independent variable x is changed to y, then the differential equation $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0$ is changed to $x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = k$, where k equals to .....

29. The curve passing through the point (1, 1) satisfies the differential equation $\frac{dy}{dx} + \frac{\sqrt{(x^2-1)(y^2-1)}}{xy} = 0$. If the curve passes through the point $(\sqrt{2}, k)$, then the value of $[k]$ is (where, $[\cdot]$ represents the greatest integer function) ..... .

30. If the solution of the differential equation $\frac{dy}{dx} - y = 1 - e^{-x}$ and $y(0) = y_0$ has a finite value, when $x \to \infty$, then the value of $\left|\frac{2}{y_0}\right|$ is ..... .

Answers with Explanation

1. ***(b)*** The given differential equation can be written as

$$\left(\frac{d^3y}{dx^3}\right)^2 - (x^2-1)\frac{d^3y}{dx^3} + \left(\frac{d^3y}{dx^3}\right)\left(\frac{d^2y}{dx^2}\right)^5 + 4\left(\frac{d^2y}{dx^2}\right)^3 = 0$$

Clearly, its order is 3 and degree is 2.

$\therefore\ m = 3$ and $n = 2$

2. ***(b)*** We have, $x = e^{\frac{dy}{dx}} \Rightarrow \log\ x = \frac{dy}{dx}$

Clearly, it is differential equation of degree 1.

3. ***(c)*** Let the equation of the family F of required ellipse be

$$Ax^2 + By^2 = 1 \qquad \ldots(i)$$

It is a two parameter family of curves.

On differentiating Eq. (i) w.r.t. x, we get

$$Ax + Byy' = 0 \qquad \ldots(ii)$$

On differentiating Eq. (ii) w.r.t. x, we get

$$A + B(y')^2 + Byy'' = 0 \qquad \ldots(iii)$$

On multiplying Eq. (iii) by x and subtracting from Eq. (ii), we get

$$B[yy' - x(y')^2 - xyy''] = 0$$

$$\Rightarrow \quad xyy'' + x(y')^2 - yy' = 0$$

$$\Rightarrow \quad xy\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(x\frac{dy}{dx} - y\right) = 0$$

4. ***(b)*** **Key point** The differential equation of the orthogonal trajectories of given equation is obtained by replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

The equation of the given family of curves is

$$a^{n-1}y = x^n \qquad \ldots(i)$$

$$\Rightarrow \quad (n-1)\log\ a + \log\ y = n\log\ x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{n}{x} \qquad \ldots(ii)$$

Which is the differential equation of the family of curves given in Eq. (i).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (ii), we get

$$\frac{1}{y}\left(-\frac{dx}{dy}\right) = \frac{n}{x} \Rightarrow\ x\,dx + ny\,dy = 0$$

On integrating both sides, we get

$$\frac{x^2}{2} + n\frac{y^2}{2} = C \Rightarrow x^2 + ny^2 = 2C$$

5. ***(a)*** The given equation can be written as

$$\frac{y}{x}\cdot\frac{dy}{dx} = \left\{\frac{y^2}{x^2} + \frac{f\left(\frac{y^2}{x^2}\right)}{f'\left(\frac{y^2}{x^2}\right)}\right\}$$

The above equation is a homogeneous equation.

Putting $y = vx$, we get

$$v\left[v + x\frac{dv}{dx}\right] = v^2 + \frac{f(v^2)}{f'(v^2)}$$

$$\Rightarrow \quad vx\frac{dv}{dx} = \frac{f(v^2)}{f'(v^2)} \Rightarrow \frac{2vf'(v^2)}{f(v^2)}dv = \frac{2}{x}dx$$

Now, integrating both sides, we get

$$\log\ f(v^2) = \log x^2 + \log\ C$$

$$\Rightarrow \quad \log\ f(v^2) = \log\ Cx^2 \Rightarrow f(v^2) = Cx^2$$

$$\Rightarrow \quad f\left(\frac{y^2}{x^2}\right) = Cx^2$$

6. ***(a)*** We have, $x^2\frac{dy}{dx}\cos\left(\frac{1}{x}\right) - y\sin\left(\frac{1}{x}\right) = -1$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x^2}\tan\left(\frac{1}{x}\right) = -\sec\left(\frac{1}{x}\right)\cdot\frac{1}{x^2} \qquad \text{[linear]}$$

$$\therefore \quad \text{IF} = e^{\int -\left(\frac{1}{x^2}\right)\tan\left(\frac{1}{x}\right)dx} = \sec\left(\frac{1}{x}\right)$$

Thus, the solution is

$$y\sec\frac{1}{x} = \int -\sec^2\frac{1}{x}\cdot\frac{1}{x^2}dx$$

$$\Rightarrow \quad y\sec\left(\frac{1}{x}\right) = \tan\left(\frac{1}{x}\right) + C$$

Given, $y \to -1, x \to \infty$. Thus, $C = -1$

Hence, the equation of curve is $y = \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$.

7. ***(b)*** We have, $x^2 = e^{\left(\frac{x}{y}\right)^{-1}\left(\frac{dy}{dx}\right)}$

$$\Rightarrow \quad x^2 = e^{\left(\frac{y}{x}\right)\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow \quad \ln x^2 = \frac{y}{x}\cdot\frac{dy}{dx} \Rightarrow \int x\ln x^2\,dx = \int y\,dy$$

Putting $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad \frac{1}{2}\int \ln t\,dt = \frac{y^2}{2}$$

$$\Rightarrow \quad C + t\ln t - t = y^2$$

$$\Rightarrow \quad y^2 = x^2\ln x^2 - x^2 + C$$

$$\Rightarrow \quad y^2 = x^2(\ln x^2 - 1) + C$$

8. ***(a)*** Let $v(t)$ be the velocity of the object at time t.

Given, $\frac{dv}{dt} = 9.8 - kv$ or $\frac{dv}{9.8 - kv} = dt$

On integrating both sides, we get

$$\log\ (9.8 - kv) = -kt + \log\ C$$

$$\Rightarrow \quad 9.8 - kv = Ce^{-kt}$$

But $v(0) = 0 \Rightarrow C = 9.8$

Thus, $9.8 - kv = 9.8e^{-kt}$

$$\Rightarrow \quad kv = 9.8(1 - e^{-kt})$$

$$\Rightarrow \quad v(t) = \frac{9.8}{k}(1 - e^{-kt}) < \frac{9.8}{k} \text{ for all } t.$$

Hence, $v(t)$ cannot exceed $\frac{9.8}{k}$ m/s.

9. ***(a)*** Let $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

Hence, the differential equation becomes

$$(e^{x^2}+e^t)\frac{dt}{dx}+2e^{x^2}(xt-x)=0$$

$$\Rightarrow \quad e^{x^2}+e^t+2e^{x^2}x(t-1)\frac{dx}{dt}=0$$

Put $e^{x^2}=z \Rightarrow e^{x^2}2x\frac{dx}{dt}=\frac{dz}{dt}$

$$\therefore \quad z+e^t+\frac{dz}{dt}(t-1)=0$$

$$\Rightarrow \quad \frac{dz}{dt}+\frac{1}{(t-1)}z=-\frac{e^t}{t-1} \qquad \text{[linear]}$$

$$\therefore \quad \text{IF}=e^{\int\frac{1}{t-1}dt}=t-1$$

Hence, the solution is $z(t-1)=\int\frac{-e^t}{t-1}\cdot(t-1)dt+C$

$$\Rightarrow \quad z(t-1)=-e^t+C \Rightarrow e^{x^2}(y^2-1)=-e^{y^2}+C$$

$$\Rightarrow \quad e^{x^2}(y^2-1)+e^{y^2}=C$$

10. ***(a)*** $\frac{dy}{dx}=\frac{1}{xy[x^2\sin y^2+1]} \Rightarrow \frac{dx}{dy}=xy[x^2\sin y^2+1]$

$$\Rightarrow \quad \frac{1}{x^3}\cdot\frac{dx}{dy}-\frac{1}{x^2}y=y\sin y^2$$

Putting $-\frac{1}{x^2}=u$, the equation can be written as

$$\frac{du}{dy}+2uy=2y\sin y^2 \qquad \text{[linear]}$$

$$\therefore \quad \text{IF}=e^{\int 2y}=e^{y^2}$$

Thus, the solution is

$$u\cdot e^{y^2}=\int 2y\sin y^2e^{y^2}\,dy+C=\int\sin y^2e^{y^2}d(y^2)+C$$

$$=\frac{1}{2}e^{y^2}(\sin y^2-\cos y^2)+C$$

$$\Rightarrow 2ue^{y^2}=e^{y^2}(\sin y^2-\cos y^2)+2C$$

$$\Rightarrow \quad 2u=\sin y^2-\cos y^2+2Ce^{-y^2}$$

$$\Rightarrow \quad 2=x^2[\cos y^2-\sin y^2-2Ce^{-y^2}]$$

11. ***(c)*** Rewriting the given equation, we get

$$2xy\frac{dy}{dx}-y^2=1+x^2 \Rightarrow 2y\frac{dy}{dx}-\frac{1}{x}y^2=\frac{1}{x}+x$$

Putting $y^2=u$, we have

$$\frac{du}{dx}-\frac{1}{x}u=\frac{1}{x}+x \qquad \text{[linear]}$$

$$\therefore \quad \text{IF}=e^{\int-\frac{1}{x}dx}=\frac{1}{x}$$

Thus, the solution is

$$u\cdot\frac{1}{x}=\int\left(\frac{1}{x^2}+1\right)dx \Rightarrow \frac{u}{x}=-\frac{1}{x}+x+C$$

$$\Rightarrow \quad y^2=(x^2-1)+Cx$$

Since, $y(1)=1$, we get $C=1$

Hence, $y^2=x^2-1+x$

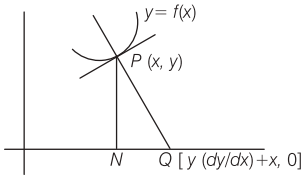
$$\Rightarrow \quad y^2=\left(x-\frac{1}{2}\right)^2-\frac{5}{4}$$

$$\Rightarrow \quad \frac{\left(x-\frac{1}{2}\right)^2}{5/4}-\frac{y^2}{5/4}=1$$

which represent a system of hyperbola.

12. ***(c)*** Equation of normal at $P(x, y)$ is

$$Y-y=-\frac{dx}{dy}(X-x)$$



Also given, $NQ=y\frac{dy}{dx}=\frac{x(1+y^2)}{1+x^2}$

$$\therefore \quad \frac{ydy}{1+y^2}=\frac{x}{1+x^2}dx$$

$$\Rightarrow \quad \ln(1+y^2)=\ln(1+x^2)+\ln C$$

$$\Rightarrow \quad 1+y^2=C(1+x^2)$$

Since, the curve passes through (3, 1).

Thus, $1+1=C(1+9) \Rightarrow C=\frac{1}{5}$

Thus, curve is $1+y^2=\frac{1}{5}(1+x^2)$ i.e. $x^2-5y^2=4$.

13. ***(a, c)*** We have,

$$(x-h)^2+(y-k)^2=a^2 \qquad \ldots(i)$$

On differentiating w.r.t. x, we get

$$2(x-h)+2(y-k)\frac{dy}{dx}=0$$

$$\Rightarrow \quad (x-h)+(y-k)\frac{dy}{dx}=0 \qquad \ldots(ii)$$

Again, differentiating w.r.t. x, we get

$$1+\left(\frac{dy}{dx}\right)^2+(y-k)\frac{d^2y}{dx}=0 \qquad \ldots(iii)$$

From Eq. (iii),

$$y-k=-\left(\frac{1+p^2}{q}\right)$$

where, $p=\frac{dy}{dx}, q=\frac{d^2y}{dx^2}$

Putting the value of $(y-k)$ in Eq. (iii), we get

$$x-h=\frac{(1+p^2)p}{q}$$

Substituting the values of $(x-h)$ and $(y-k)$ in Eq. (i), we get

$$\left(\frac{1+p^2}{q}\right)^2(1+p^2)=a^2 \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

14. ***(a, b, c)*** By using basic concepts of differential equation.

15. ***(b, c)*** Given, $(x^2y^2-1)\,dy+2xy^3dx=0$

$$\Rightarrow \quad x^2y^2dy+2xy^3dx=dy \Rightarrow x^2dy+2xy\,dx=\frac{dy}{y^2}$$

$$\Rightarrow \quad \int d(x^2y)=\int\frac{dy}{y^2}+C \Rightarrow x^2y=\frac{-1}{y}+C$$

$$\Rightarrow \quad x^2y+\frac{1}{y}=C \quad \Rightarrow x^2y^2+1=Cy$$

16. ***(a, d)*** The differential equation can be rewritten as

$$\frac{xdx + ydy}{\sqrt{1-(x^2+y^2)}} = \frac{xdy - ydx}{\sqrt{x^2+y^2}}$$

Since, $d\tan^{-1}(y/x) = \dfrac{xdy - ydx}{x^2+y^2}$

and $d(x^2+y^2) = 2(xdx + ydy)$, we have

$$\frac{\frac{1}{2}d(x^2+y^2)}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}} = \frac{xdy - ydx}{x^2+y^2} = d\{\tan^{-1}(y/x)\}$$

Putting $x^2+y^2 = t^2$ in the LHS, we get

$$\frac{tdt}{t\sqrt{1-t^2}} = d\{\tan^{-1}(y/x)\}$$

Integrating both sides, we get

$$\sin^{-1}t = \tan^{-1}(y/x) + C$$

$$\Rightarrow \quad \sin^{-1}\sqrt{x^2+y^2} = \tan^{-1}(y/x) + C$$

17. ***(a, b)*** (c) $x = \sin\left(\dfrac{dy}{dx} - 2y\right) \Rightarrow \dfrac{dy}{dx} - 2y = \sin^{-1}x$

(d) $x - 2y = \log\left(\dfrac{dy}{dx}\right) \Rightarrow \dfrac{dy}{dx} = e^{x-2y}$

In both cases, degree is defined as it is 1.

18. ***(c)*** Given equation can be rewritten as

$$y = xp + \sqrt{1+p^2},\ p = \frac{dy}{dx}$$

On differenting w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + \frac{1}{2\sqrt{1+p^2}}2p\cdot\frac{dp}{dx}$$

$$\frac{dp}{dx} = 0 \text{ or } \frac{p}{\sqrt{1+p^2}} = -x, \quad p = C \text{ or } p = \frac{x}{\sqrt{1-x^2}}$$

Thus, $y = cx + \sqrt{(1+c^2)}$ gives the general solution and $x^2+y^2 = 1$ as singular solution.

19. ***(b)*** $y = xp + p^2 \quad \left[\text{where } p = \dfrac{dy}{dx}\right] \ldots(i)$

On differentiating Eq. (i) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}(x+2p) = 0 \Rightarrow \frac{dp}{dx} = 0 \text{ or } p = -\frac{x}{2}$$

Eliminating p from Eq. (i), we get $y = -\dfrac{x^2}{2} + \dfrac{x^2}{4} = -\dfrac{x^2}{4}$

Clearly, $f(x) = -\dfrac{x^2}{4} = -1$ has two solutions.

20. ***(d)*** Given, $y = mx + m - m^3 \quad \ldots(i)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$\Rightarrow m = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx} \quad \left[\because \frac{dy}{dx} = m\,(\text{given})\right]$$

$$\frac{dm}{dx}(x+1-3m^2) = 0 \Rightarrow \frac{dm}{dx} = 0 \text{ or } m = C \quad \ldots(ii)$$

$$\Rightarrow \quad x + 1 - 3m^2 = 0 \text{ or } m^2 = \frac{x+1}{3} \quad \ldots(iii)$$

Eliminating m between Eqs. (i) and (iii), we get

$$y = m(x+1-m^2) = \left(\frac{x+1}{3}\right)^{1/2}\left(x+1-\frac{x+1}{3}\right)$$

$$\Rightarrow \quad y = \left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}(x+1) = 2\left(\frac{x+1}{3}\right)^{3/2}$$

$$\Rightarrow \quad y^2 = \frac{4}{27}(x+1)^3 \Rightarrow 27y^2 = 4(x+1)^3$$

which passes through $(-1, 0)$.

Sol. (Q. Nos. 21-23)

Integrating $\dfrac{d^2y}{dx^2} = 6x - 4$, we get $\dfrac{dy}{dx} = 3x^2 - 4x + A$

When $x = 1$, then $\dfrac{dy}{dx} = 0$, so that $A = 1$.

Hence, $\dfrac{dy}{dx} = 3x^2 - 4x + 1 \quad \ldots(i)$

Integrating, we get $y = x^3 - 2x^2 + x + B$

When $x = 1$, then $y = 5$, so that $B = 5$

Thus, we have $y = x^3 - 2x^2 + x + 5$

From Eq. (i), we get the critical points $x = 1/3, x = 1$

At the critical point $x = \dfrac{1}{3}$, $\dfrac{d^2y}{dx^2}$ is negative

Therefore, at $x = 1/3$, y has a local maximum.

At $x = 1$, $\dfrac{d^2y}{dx^2}$ is positive.

Therefore, at $x = 1$, y has a local minimum.

Also, $f(1) = 5$, $f\left(\dfrac{1}{3}\right) = \dfrac{139}{27}$, $f(0) = 5$, $f(2) = 7$

Hence, the global maximum value $= 7$ and the global minimum value $= 5$.

21. ***(c)*** **22.** ***(a)*** **23.** ***(b)***

24. A $\rightarrow$ q, s; B $\rightarrow$ p; C $\rightarrow$ p; D $\rightarrow$ q, r, s

A. Equation of the required parabola is of the form $y^2 = 4a(x-h)$.

Differentiating w.r.t. x, we have

$$2y\frac{y}{dx} = 4a \text{ or } y\frac{dy}{dx} = 2a \text{ or } \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$$

The degree of this differential equation is 1 and the order is 2.

B. We have, $y = a(x+a)^2 \quad \ldots(i)$

$$\therefore \quad \frac{dy}{dx} = 2a(x+a) \quad \ldots(ii)$$

Dividing Eq. (i) by Eq.(ii), we get

$$\frac{y}{\frac{dy}{dx}} = \frac{x+a}{2} \Rightarrow x + a = \frac{2y}{y_1}, \text{ where } y_1 = \frac{dy}{dx}$$

Substituting $a = \dfrac{2y}{y_1} - x$ in Eq. (i), we get

$$y = \left(\frac{2y}{y_1} - x\right)\left(\frac{2y}{y_1}\right)^2 \text{ or } y_1^3 y = 4(2y - xy_1)\,y^2$$

Clearly, it is a differential equation of degree 3.

C. The given equation is $\left(1 + 3\dfrac{dy}{dx}\right)^{2/3} = 4\dfrac{d^3y}{dx^3}$

Cubing, we get $\left(1+3\frac{dy}{dx}\right)^2 = 64\left(\frac{d^3y}{dx^3}\right)^3$

Hence, order = degree = 3

D. We have, $y^2 = (x+\sqrt{c})$

On differenting w.r.t. x, we get

$$2y\frac{dy}{dx} = 2c \text{ or } c = y\frac{dy}{dx}$$

Putting in Eq. (i), we get

$$y^2 = 2\left(y\frac{dy}{dx}\right)x + 2\left(y\frac{dy}{dx}\right)^{3/2}$$

$$\Rightarrow \left(y^2 - 2xy\frac{dy}{dx}\right)^2 = 4y^3\left(\frac{dy}{dx}\right)^3$$

Its order is 1 and degree is 3.

25. A → q; B → r; C → p; D → s

A. $y = e^{4x} + 2e^{-x}$, $y_1 = 4e^{4x} - 2e^{-x}$, $y_2 = 16e^{4x} + 2e^{-x}$

$y_3 = 64e^{4x} - 2e^{-x}$

Now, $y_3 - 13y_1 = (64e^{4x} - 2e^{-x}) - 13(4e^{4x} - 2e^{-x})$

$\Rightarrow \quad y_3 - 13y_1 = 12e^{4x} + 24e^{-x}$

$\Rightarrow \quad y_3 - 13y_1 = 12(e^{4x} + 2e^{-x}) = 12y$

$\therefore \; k = 12$ and $\frac{k}{3} = 4$

B. Since, equation is of degree 2, two lines are possible.

C. $y = u^m$ or $\frac{dy}{dx} = mu^{m-1}\frac{du}{dx}$

Substituting the value of y and $\frac{dy}{dx}$ in

$2x^4 y\frac{dy}{dx} + y^4 = 4x^6$, we have

$$2x^4 u^m mu^{m-1}\frac{du}{dx} + u^{4m} = 4x^6 \text{ or } \frac{du}{dx} = \frac{4x^6 - u^{4m}}{2mx^4 u^{2m-1}}$$

For homogeneous equation, $4m = 6$ or $m = \frac{3}{2}$

and $2m - 1 = 2$ or $m = \frac{3}{2}$

D. $y = Ax^m + Bx^{-n}$

$\therefore \frac{dy}{dx} = Amx^{m-1} - nBx^{-n-1}$

Now, $\frac{d^2y}{dx^2} = Am(m-1)x^{m-2} + n(n+1)Bx^{-n-2}$

Putting these values in $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 12y$, we get

$m(m+1)Ax^m + n(n-1)Bx^{-n} = 12(Ax^m + Bx^{-n})$

i.e. $m(m+1) = 12$ or $n(n-1) = 12$

i.e. $m = 3, -4$ or $n = 4, -3$

26. *(2)* Given, $\frac{dy}{dx} - \frac{1}{x}y = \left(x - \frac{2}{x}\right)$

$$\therefore \text{IF} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Now, the general solution is given by

$$\frac{y}{x} = \int\left(x - \frac{2}{x}\right)\frac{1}{x}dx \text{ or } \frac{y}{x} = x + \frac{2}{x} + C$$

As $y(1) = 1, C = -2$

$$\therefore \quad \frac{y}{x} = x + \frac{2}{x} - 2 \text{ or } y = x^2 - 2x + 2$$

Hence, $y(2) = (2)^2 - 2(2) + 2 = 2$

27. *(8)* Given, $\frac{dy}{dt} + 2ty = t^2$

$$\therefore \quad \text{IF} = e^{t^2}$$

Thus, solution is $y \cdot e^{t^2} = \int t^2 \cdot e^{t^2}dt = \frac{1}{2}\int t(2t \cdot e^{t^2})\,dt$

$$\therefore \quad y \cdot e^{t^2} = t \cdot \frac{e^{t^2}}{2} - \frac{1}{2}\int e^{t^2}dt + C$$

$$\Rightarrow \quad y = \frac{t}{2} - e^{-t^2}\int\frac{e^{t^2}}{2}dt + Ce^{-t^2}$$

$$\therefore \quad \lim_{t\to\infty}\frac{y}{t} = \frac{1}{2} - \lim_{t\to\infty}\frac{\int\frac{e^{t^2}}{2}}{te^{t^2}} + \frac{C}{1\cdot e^{t^2}} = \frac{1}{2}$$

28. *(1)* $\frac{dy}{dx} = \frac{1}{dx/dy} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{d}{dx/dy}\right)\cdot\frac{dy}{dx}$

$$= -\frac{1}{(dx/dy)^3}\cdot\frac{d^2x}{dy^2}$$

Hence, $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \left(\frac{dy}{dx}\right) = 0$

Becomes $(-x)\cdot\frac{1}{(dx/dy)^3}\cdot\frac{d^2x}{dy^2} + \frac{1}{(dx/dy)^3} - \frac{1}{(dx/dy)} = 0$

$$\Rightarrow \quad x\frac{d^2x}{dy^2} - 1 + \left(\frac{dx}{dy}\right)^2 = 0 \quad \Rightarrow \quad x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = 1$$

$$\therefore \quad k = 1$$

29. *(3)* $\frac{dy}{dx} = -\frac{\sqrt{(x^2-1)(y^2-1)}}{xy}$

$$\int\frac{y}{\sqrt{y^2-1}}dy = -\int\frac{\sqrt{x^2-1}}{x}dx$$

Put $\quad y^2 - 1 = t^2 \Rightarrow 2y\,dy = 2t\,dt$

$$\therefore \quad \int\frac{t}{t}dt = -\int\frac{x^2-1}{x\sqrt{x^2-1}}dx$$

$$\Rightarrow \quad t = -\int\frac{x}{\sqrt{x^2-1}}dx + \int\frac{1}{x\sqrt{x^2-1}}dx$$

$$\therefore \quad \sqrt{y^2-1} = -\sqrt{x^2-1} + \sec^{-1}x + C$$

Since, curve passes through the point (1, 1).

Then, the value of $C = 0$.

Hence, the curve is $\sqrt{y^2-1} = -\sqrt{x^2-1} + \sec^{-1}x$.

30. *(4)* $\frac{dy}{dx} - y = 1 - e^{-x}$, $\quad P = -1, Q = 1 - e^{-x}$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int -1dx} = e^{-x}$$

$$\therefore \; y\cdot e^{-x} = \int e^{-x}(1 - e^{-x})\,dx + C = -e^{-x} + \frac{1}{2}e^{-2x} + C$$

$$\Rightarrow \quad y = -1 + \frac{1}{2}e^{-x} + Ce^{x}$$

$$\therefore \quad x = 0, y = y_0$$

So, $\quad C = y_0 + \frac{1}{2}, \quad y = -1 + \frac{1}{2}e^{-x} + \left(y_0 + \frac{1}{2}\right)e^x$

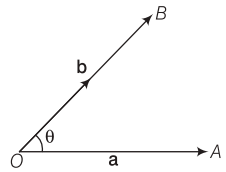
$x \to \infty, y \to$ Finite value

So, $\quad y_0 + \frac{1}{2} = 0 \Rightarrow y_0 = -\frac{1}{2}$

CONCE your Revi VECTOR

❶ THE SCALAR OR DOT PRODUCT

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors inclined at an angle θ. Then, the scalar product of $\mathbf{a}$ with $\mathbf{b}$ is denoted by $\mathbf{a}\cdot\mathbf{b}$ and is defined as the scalar $|\mathbf{a}||\mathbf{b}|\cos\theta$.



Thus, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$

REMARK If $\mathbf{a}$ or $\mathbf{b}$ or both is a zero vector, then θ is not defined as 0 has no direction. In this case their dot product $\mathbf{a}\cdot\mathbf{b}$ is defined as the scalar zero.

❷ PROJECTION

The projection of a vector is the scalar product of two vectors with the product of modulus of either vector and the projection of the other in its direction.

$\therefore$ Projection of $\mathbf{b}$ on $\mathbf{a}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=\dfrac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}=\hat{\mathbf{a}}\cdot\mathbf{b}$

and projection of $\mathbf{a}$ on $\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}=\mathbf{a}\cdot\hat{\mathbf{b}}$

❸ PROPERTIES OF SCALAR PRODUCT

Ⓐ **Property I** The scalar product of two vectors is commutative, i.e. $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$.

Ⓑ **Property II** The scalar product of vectors is distributive over vector addition, i.e.

(i) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ [left distributive]

(ii) $(\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}$ [right distributive]

Ⓒ **Property III** Let $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors. Then, $\mathbf{a}\cdot\mathbf{b}=0\Leftrightarrow\mathbf{a}\perp\mathbf{b}$

REMARK Since, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are mutually perpendicular unit vectors along the coordinate axes. Therefore,

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{i}}=0,\ \hat{\mathbf{j}}\cdot\hat{\mathbf{k}}=\hat{\mathbf{k}}\cdot\hat{\mathbf{j}}=0,\ \hat{\mathbf{k}}\cdot\hat{\mathbf{i}}=\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}=0$$

Ⓓ **Property IV** For any vector $\mathbf{a}$, we have $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$

REMARK Since, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors along the coordinate axes. Therefore,

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}=|\hat{\mathbf{i}}|^2=1,\ \hat{\mathbf{j}}\cdot\hat{\mathbf{j}}=|\hat{\mathbf{j}}|^2=1 \text{ and } \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=|\hat{\mathbf{k}}|^2=1$$

Ⓔ **Property V** If m is a scalar and $\mathbf{a}$, $\mathbf{b}$ are any two vectors, then

$$(m\cdot\mathbf{a})\cdot\mathbf{b}=m(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(m\cdot\mathbf{b})$$

Ⓕ **Property VI** If m and n are scalars and $\mathbf{a}$, $\mathbf{b}$ are two vectors, then

$$m\mathbf{a}\cdot n\mathbf{b}=mn(\mathbf{a}\cdot\mathbf{b})=(mn\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(mn\mathbf{b})$$

Ⓖ **Property VII** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(i) $\mathbf{a}\cdot(-\mathbf{b})=-(\mathbf{a}\cdot\mathbf{b})=(-\mathbf{a})\cdot\mathbf{b}$

(ii) $(-\mathbf{a})\cdot(-\mathbf{b})=\mathbf{a}\cdot\mathbf{b}$

Ⓗ **Property VIII** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(i) $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b}$

(ii) $|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}$

(iii) $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=|\mathbf{a}|^2-|\mathbf{b}|^2$

(iv) $|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2=2(|\mathbf{a}|^2+|\mathbf{b}|^2)$

(v) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|\Leftrightarrow\mathbf{a}\perp\mathbf{b}$

(vi) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}|+|\mathbf{b}|\Leftrightarrow\mathbf{a}$ is parallel to $\mathbf{b}$.

(vii) $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2$

$\Leftrightarrow\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

Ⓘ **Property IX** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors, then

$$|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$$

❹ SCALAR PRODUCT IN TERMS OF COMPONENTS

Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ and $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$.

Then, by using the distributivity of dot products over vector addition, we get

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$$

❺ ANGLE BETWEEN TWO VECTORS

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors inclined at an angle θ.

Then, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$

$$\Rightarrow\quad\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}$$

$$\Rightarrow\quad\theta=\cos^{-1}\left[\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}\right]$$

If $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$

and $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$,

then $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$

$$|\mathbf{a}|=\sqrt{a_1^2+a_2^2+a_3^2},\ |\mathbf{b}|=\sqrt{b_1^2+b_2^2+b_3^2}$$

$$\theta=\cos^{-1}\left[\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right]$$

❻ COMPONENTS OF A VECTOR ALONG AND PERPENDICULAR TO ANOTHER VECTOR

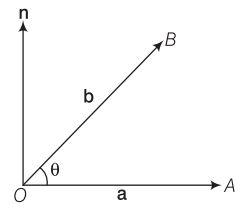
The component of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are

$$\left\{\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right\}\mathbf{a}$$

and $\mathbf{b}-\left\{\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right\}\mathbf{a}$, respectively.

❼ VECTOR OR CROSS PRODUCT

Let $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-parallel vectors. Then, the vector product is defined as $\mathbf{a}\times\mathbf{b}$ in order as a vector whose magnitude is $|\mathbf{a}||\mathbf{b}|\sin\theta$.



VECTOR TRIPLE PRODUCT

For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the vectors $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ are called vector triple products of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

If any three vectors are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

AID TO MEMORY

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ is written as

$\text{I}\times(\text{II}\times\text{III})=(\text{I}\cdot\text{III})\,\text{II}-(\text{I}\cdot\text{II})\,\text{III}$

PROPERTIES OF VECTOR TRIPLE PRODUCT

1. The vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a linear combination of those two vectors which are within brackets.

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and whose direction is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ in such a way that $\mathbf{a}$, $\mathbf{b}$ and this direction constitute a right handed system.

$\therefore\quad\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

where, $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$.

Note
- If one of $\mathbf{a}$ or $\mathbf{b}$ or both is $\mathbf{0}$, then θ is not defined as $\mathbf{0}$ has no direction and so $\hat{\mathbf{n}}$ is not defined. In such a case, $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.
- If $\mathbf{a}$ and $\mathbf{b}$ are collinear, i.e. if $\theta=0$ or π, then the direction of $\hat{\mathbf{n}}$ is not well-defined and so in this case also $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.

❽ PROPERTIES OF VECTOR PRODUCT

Ⓐ **Property I** Vector product is not commutative, i.e. $\mathbf{a}\times\mathbf{b}\neq\mathbf{b}\times\mathbf{a}$

However, $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$

Ⓑ **Property II** If $\mathbf{a}$ and $\mathbf{b}$ are two vectors and m is a scalar, then

$$m\cdot\mathbf{a}\times\mathbf{b}=m(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times m\mathbf{b}$$

Ⓒ **Property III** If $\mathbf{a}$ and $\mathbf{b}$ are two vectors and m, n are scalars, then

$$m\mathbf{a}\times n\mathbf{b}=mn(\mathbf{a}\times\mathbf{b})=m(\mathbf{a}\times n\mathbf{b})=n(m\mathbf{a}\times\mathbf{b})$$

Ⓓ **Property IV** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be any three vectors, then

(i) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ [left distributivity]

(ii) $(\mathbf{b}+\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}$ [right distributivity]

Ⓔ **Property V** For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we have

$$\mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}$$

Ⓕ **Property VI** The vector product of two non-zero vectors is zero vector iff they are parallel (collinear),

PT MAP sion Tool PRODUCT

2. The vector $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane $\mathbf{b}$ and $\mathbf{c}$.
3. The formula $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is true only when the vector outside the bracket is on the left-most side. If it is not we first shift on left by using the properties of cross product and then apply the same formula.
$$(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\}$$
$$= -\{(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\}$$
$$= -(\mathbf{a} \cdot \mathbf{c})\mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$
4. For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the value of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ is zero. Thus, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$, $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ are coplanar.

i.e. $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \,||\, \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

REMARK It follows from the above property that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every non-zero vector $\mathbf{a}$.

Ⓖ **Property VII** Vector product of orthogonal triad of unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$

$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$

$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$

Ⓗ **Property VIII** Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ be two vectors.

Then, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Ⓘ **Property IX** Unit vectors normal to the plane containing two non-zero, non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ are given by $\pm \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$.

Ⓙ **Property X** The area of parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.

Ⓚ **Property XI** The area of a triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

Ⓛ **Property XII** The area of a ΔABC is $\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|$ or $\frac{1}{2}|\mathbf{BC} \times \mathbf{BA}|$ or $\frac{1}{2}|\mathbf{CB} \times \mathbf{CA}|$.

Ⓜ **Property XIII** The area of a parallelogram with diagonals $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

Ⓝ **Property XIV** The area of a plane quadrilateral $ABCD$ is $\frac{1}{2}|\mathbf{AC} \times \mathbf{BD}|$, where $\mathbf{AC}$ and $\mathbf{BD}$ are its diagonals.

⑨ SCALAR TRIPLE PRODUCT

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors. Then, the scalar $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called the scalar triple product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and is denoted by $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Thus, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

The scalar triple product $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ represents the volume of the parallelopiped whose coterminous edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right handed system of vectors.

⑩ PROPERTIES OF SCALAR TRIPLE PRODUCT

Ⓐ **Property I** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are cyclically permuted, then value of scalar triple product remains same, i.e. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$

or $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

Ⓑ **Property II** The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude, i.e.
$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$$
$$= -[\mathbf{c}\ \mathbf{b}\ \mathbf{a}]$$
$$= -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$$

Ⓒ **Property III** In scalar triple product, the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remain same, i.e.
$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Ⓓ **Property IV** The scalar triple product of three vectors is zero, if any two of them are equal.

i.e. $[\mathbf{a}\ \mathbf{a}\ \mathbf{c}] = 0$

Ⓔ **Property V** For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and scalars λ, μ and ν, we have
$$[\lambda\mathbf{a}\ \mu\mathbf{b}\ \nu\mathbf{c}] = \lambda\mu\nu\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

Ⓕ **Property VI** The scalar triple product of three vectors is zero, if any two of them are parallel or collinear.

Ⓖ **Property VII** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are four vectors, then
$$[\mathbf{a} + \mathbf{b}\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$$

Ⓗ **Property VIII** Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$,
$$\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$$
and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$
be the three vectors.
Then, we have
$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Ⓘ **Property IX** The necessary and sufficient condition for three non-zero, non-collinear vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to be coplanar,

i.e. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

Ⓙ **Property X** Four points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ will be coplanar, if
$$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

Ⓚ **Property XI** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three non-coplanar vectors and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are three vectors such that
$$\mathbf{A} = x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c},$$
$$\mathbf{B} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$$
and $\mathbf{C} = x_3\mathbf{a} + y_3\mathbf{b} + z_3\mathbf{c}$, then
$$\therefore \quad [\mathbf{A}\,\mathbf{B}\,\mathbf{C}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

Ⓛ **Property XII** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are two trials of vectors, then
$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{l}\ \mathbf{m}\ \mathbf{n}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{l} & \mathbf{b}\cdot\mathbf{l} & \mathbf{c}\cdot\mathbf{l} \\ \mathbf{a}\cdot\mathbf{m} & \mathbf{b}\cdot\mathbf{m} & \mathbf{c}\cdot\mathbf{m} \\ \mathbf{a}\cdot\mathbf{n} & \mathbf{b}\cdot\mathbf{n} & \mathbf{c}\cdot\mathbf{m} \end{vmatrix}$$

Ⓜ **Property XIII** For any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,
$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

Ⓝ **Property XIV** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{l}$, $\mathbf{m}$ are vectors, then
$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,(\mathbf{l} \times \mathbf{m}) = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{l} & \mathbf{b}\cdot\mathbf{l} & \mathbf{c}\cdot\mathbf{l} \\ \mathbf{a}\cdot\mathbf{m} & \mathbf{b}\cdot\mathbf{m} & \mathbf{c}\cdot\mathbf{m} \end{vmatrix}$$

⑪ VOLUME OF TETRAHEDRON

The volume of a tetrahedron whose three coterminous edges in the right handed system are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $\frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

If A, B, C, D are the vertices of a tetrahedron $ABCD$, then its volume is numerically equal to
$$\frac{1}{6}[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}]$$
or $$\frac{1}{6}[\mathbf{BC}\ \mathbf{BD}\ \mathbf{BA}]$$
or $$\frac{1}{6}[\mathbf{CD}\ \mathbf{CB}\ \mathbf{CA}].$$

RECIPROCAL SYSTEM OF VECTORS

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-coplanar vectors and let
$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},$$
$$\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$
and $$\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$
where, $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are said to form a reciprocal system of vectors for the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Love Agarwal

CUBE ROOTS AND *n*th ROOTS OF UNITY

*n*TH ROOTS OF UNITY IS VERY IMPORTANT TOPIC OF COMPLEX NUMBER. SPECIALLY CUBE ROOTS OF UNITY (ω) IS ONE OF THE MOST IMPORTANT TOPIC FROM WHICH EVERY YEAR ATLEAST ONE QUESTION IS ASKED IN IIT JEE MAIN. THIS SECTION WILL COVER BASIC CONCEPT AND VARIOUS PROPERTIES OF CUBE ROOTS OF UNITY.

*n*th Roots of a Complex Number

Let $z = a + ib$ be a complex number and let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, by De-Moivre's theorem,

$$z^{1/n} = r^{1/n}\left\{\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right\}$$

where, $z^{1/n}$ has n distinct values.

Algorithm to Find the *n*th Roots of a Complex Number

Algorithm

***Step* I** Write the given complex number in polar form.

***Step* II** Add $2r\pi$ to the argument.

***Step* III** Apply De-Moivre's theorem.

***Step* IV** Put $r = 0, 1, 2, \ldots, (n-1)$, i.e. one less than the number in the denominator of the given index in the lowest form.

Example 1 The number of roots of the equation $z^6 = -64$ whose real parts are non-negative, is

(a) 2 *(b)* 3 *(c)* 4 *(d)* 5

Sol. ***(c)*** We have, $z^6 = -64$

$$\Rightarrow \quad z = (-64)^{1/6}$$

$$\Rightarrow \quad z = 2(-1)^{1/6}$$

$$\Rightarrow \quad z = 2(\cos\pi + i\sin\pi)^{1/6}$$

$$\Rightarrow \quad z = 2\{\cos(2r\pi + \pi) + i\sin(2r\pi + \pi)\}^{1/6}, r \in Z$$

$$\Rightarrow z = 2\left\{\cos(2r+1)\frac{\pi}{6} + i\sin(2r+1)\frac{\pi}{6}\right\}; r = 0,1,2,3,4,5$$

$$\Rightarrow z = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right), 2i, 2\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right), 2\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right),$$

$$-2i, 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

Clearly, four of these complex numbers have non-negative real parts.

POINT TO REMEMBER

De-Moivre's Theorem

If $n \in Z$, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

nth Roots of Unity

Let $z = 1^{1/n} \Rightarrow z = (\cos 0 + i\sin 0)^{1/n}$

$\Rightarrow z = (\cos 2r\pi + i\sin 2r\pi)^{1/n}$, where $r \in Z$

$\Rightarrow z = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$, where $r = 0, 1, 2, \ldots, (n-1)$

$\Rightarrow z = e^{\frac{i2r\pi}{n}}; r = 0, 1, 2, \ldots, (n-1)$ $[\because e^{i\theta} = \cos\theta + i\sin\theta]$

$\Rightarrow z = (e^{i2\pi/n})^r; r = 0, 1, 2, \ldots, (n-1)$

$\Rightarrow z = \alpha^r$, where $\alpha = e^{\frac{i2\pi}{n}}$

Thus, nth roots of unity are

$$1 (= \alpha^0), \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1},$$

where $\alpha = e^{\frac{i2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$

Example 2 If z_1 and z_2 are two nth roots of unity, then $\arg\left(\frac{z_1}{z_2}\right)$ is a multiple of

(a) $n\pi$ (b) $\frac{3\pi}{n}$ (c) $\frac{2\pi}{n}$ (d) None of these

Sol. **(c)** We have, $z_1 = e^{\frac{i2r\pi}{n}}$ and $z_2 = e^{\frac{i2s\pi}{n}}$

where, r and s are integers lying between 0 and n including 0.

$\therefore \arg\left(\frac{z_1}{z_2}\right) = \frac{2r\pi}{n} - \frac{2s\pi}{n} = 2(r-s)\frac{\pi}{n}$

Clearly, $\arg\left(\frac{z_1}{z_2}\right)$ is a multiple of $\frac{2\pi}{n}$.

Properties of nth Roots of Unity

Following are the properties of nth roots of unity

Property 1

nth roots of unity form a GP with common ratio $e^{i2\pi/n}$.

Property 2

Sum of nth roots of unity is always zero.

Let nth roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

$\therefore 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1}$

$= \frac{1-\alpha^n}{1-\alpha} = \frac{1-e^{i2\pi}}{1-\alpha}$ $\left[\because \alpha = e^{\frac{i2\pi}{n}} \Rightarrow \alpha^n = e^{i2\pi}\right]$

$= \frac{1-1}{1-\alpha} = 0$ $[\because e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1]$

Example 3 If α is a non-zero fourth root of unity, then the values of $\alpha^{4n-1} + \alpha^{4n-2} + \alpha^{4n-3}$, $n \in N$ is

(a) 0 (b) –1 (c) 3 (d) None of these

Sol. **(b)** $\alpha^{4n-1} + \alpha^{4n-2} + \alpha^{4n-3} = \alpha^{4n}(\alpha^{-1} + \alpha^{-2} + \alpha^{-3})$

$= \alpha^{-1} + \alpha^{-2} + \alpha^{-3}$ $[\because \alpha^{4n} = 1]$

$= \alpha^3 + \alpha^2 + \alpha$ $[\because \alpha^4 = 1]$

$= (1 + \alpha + \alpha^2 + \alpha^3) - 1$

$= 0 - 1$ [$\because$ sum of nth roots of unity is always zero]

$= -1$

Property 3

Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.

Let nth roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

$\therefore 1 + \alpha^p + (\alpha^2)^p + (r^3)^p + \ldots + (\alpha^{n-1})^p$

$= 1 + \alpha^p + \alpha^{2p} + r^{3p} + \ldots + \alpha^{(n-1)p}$

$= \frac{1-(\alpha^p)^n}{1-\alpha^p} = \frac{1-(e^{i2\pi p/n})^n}{1-e^{i2\pi p/n}}$

$= \frac{1-e^{i2\pi p}}{1-e^{i2\pi p/n}} = \frac{1-1}{1-e^{i2\pi p/n}}$

$[\because e^{i2\pi p} = \cos 2\pi p + i\sin 2\pi p = 1]$

$= 0$

Property 4

Sum of pth powers of nth roots of unity is n, if p is a multiple of n.

Let $p = \lambda n$, where $\lambda \in N$.

Then, $\alpha^p = e^{\frac{i2\pi p}{n}} = e^{\frac{i2\pi\lambda n}{n}} = e^{i2\pi\lambda}$

$= \cos 2\pi\lambda + i\sin 2\pi\lambda = 1$

$\therefore$ Sum of pth powers of nth roots of unity

$= 1 + \alpha^p + \alpha^{2p} + \ldots + \alpha^{(n-1)p}$

$= 1 + 1 + 1 + \ldots + 1 = n$

Property 5

Product of nth roots of unity is $(-1)^{n-1}$.

$\therefore$ Product $= 1 \cdot \alpha \cdot \alpha^2 \ldots \alpha^{n-1}$

$= \alpha^{1+2+3+\ldots+(n-1)} = \alpha^{n(n-1)/2} = (\alpha^{n/2})^{n-1}$

$= (e^{in})^{n-1} = (\cos\pi + i\sin\pi)^{n-1} = (-1)^{n-1}$

Example 4 If $z_r = \cos\left(\frac{2r\pi}{5}\right) + i\sin\left(\frac{2r\pi}{5}\right)$; $r = 0, 1, 2, 3, 4$, then $z_0z_1z_2z_3z_4$ is equal to

(a) –1 (b) 0 (c) 1 (d) None of these

Sol. **(c)** Clearly, z_0, z_1, z_2, z_3, z_4 are fifth roots of unity. We know that the product of nth roots of unity is $(-1)^{n-1}$.

Therefore, $z_0z_1z_2z_3z_4 = (-1)^{5-1} = 1$

Cube Roots of Unity

Let $z = 1^{1/3}$

Then, $z^3 = 1 \Rightarrow z^3 - 1 = 0$

$\Rightarrow (z-1)(z^2+z+1) = 0$

$\Rightarrow z - 1 = 0$ or $z^2 + z + 1 = 0$

$\Rightarrow z = 1$ or $z = \dfrac{-1 \pm \sqrt{3}i}{2}$

So, the cube roots of unity are 1, $(-1+\sqrt{3}i)/2$ and $(-1-\sqrt{3}i)/2$. Clearly, one of the roots of unity is real and other two are complex.

Note If we denote $\omega = \dfrac{-1+\sqrt{3}i}{2}$, then $\omega^2 = \dfrac{-1-\sqrt{3}i}{2}$.

Therefore, the cube roots of unity are $1, \omega, \omega^2$.

Example 5 If ω is an imaginary cube root of unity, then the value of the expression $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \ldots + (n-1)(n-\omega)(n-\omega^2)$ is [IIT JEE 2001]

(a) $\left\{\dfrac{n(n+1)}{2}\right\}^2$ (b) $\left\{\dfrac{n(n+1)}{2}\right\}^2 - n$

(c) $\left\{\dfrac{n(n+1)}{2}\right\}^2 + n$ (d) None of these

Sol. **(b)** We have,

$$(z-1)(z-\omega)(z-\omega^2) = z^3 - 1$$

$$\therefore 1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \ldots + (n-1)(n-\omega)(n-\omega^2)$$

$$= \sum_{r=2}^{n}(r-1)(r-\omega)(r-\omega^2)$$

$$= \sum_{r=2}^{n}(r^3-1) = \sum_{r=2}^{n}r^3 - \sum_{r=2}^{n}1 = \sum_{r=1}^{n}r^3 - 1 - \sum_{r=2}^{n}1$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - 1 - (n-1) = \left\{\frac{n(n+1)}{2}\right\}^2 - n$$

Properties of Cube Roots

Following are the properties of the cube root of unity

(i) Cube roots of unity are $1, \omega$ and ω^2, where $\omega = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2} = e^{i2\pi/3}$.

(ii) $\arg(\omega) = \dfrac{2\pi}{3}$ and $\arg(\omega^2) = \dfrac{4\pi}{3}$

(iii) $z^3 - 1 = (z-1)(z-\omega)(z-\omega^2)$

(iv) ω and ω^2 are roots of the equation $z^2 + z + 1 = 0$.

(v) Cube roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into three equal parts.

(vi) $1 + \omega^n + \omega^{2n} = \begin{cases} 0, & \text{if } n \text{ is not a multiple of 3} \\ 3, & \text{if } n \text{ is a multiple of 3} \end{cases}$

(vii) Cube roots of -1 are $-1, -\omega, -\omega^2$.

(viii) $z^3 + 1 = (z+1)(z+\omega)(z+\omega^2)$

(ix) $-\omega$ and $-\omega^2$ are roots of $z^2 - z + 1 = 0$.

(x) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

$a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$

$a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

Example 6 If ω is an imaginary cube root of unity, then the $(1 + \omega - \omega^2)^7$ equals [IIT JEE 1998]

(a) 128ω (b) -128ω (c) $128\omega^2$ (d) $-128\omega^2$

Sol. **(d)** $(1+\omega-\omega^2)^7 = (-2\omega^2)^7 = -128\omega^2$

Example 7 If $z(2-2\sqrt{3}i)^2 = i(\sqrt{3}+i)^4$, then $\arg(z)$ is equal to

(a) $\dfrac{5\pi}{6}$ (b) $-\dfrac{\pi}{6}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{7\pi}{6}$

Sol. **(b)** We have, $z(2-2\sqrt{3}i)^2 = i(\sqrt{3}+i)^4$

$\Rightarrow 4z(1-i\sqrt{3})^2 = i^5(1-i\sqrt{3})^4$

$\Rightarrow 4z = i(1-i\sqrt{3})^2$

$\Rightarrow 4z = i(-2\omega)^2$

$\Rightarrow 4z = 4i\omega^2$

$\Rightarrow z = i\omega^2 = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

Clearly, $\arg(z) = \dfrac{-\pi}{6}$

Example 8 Let $\omega (\neq 1)$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 2\omega^2)^{4n+3} = 0$, then the set of possible value(s) of n is(are) [JEE Advanced 2015]

(a) N (b) $\{3k : k \in N\}$

(c) $N - \{3k : k \in N\}$ (d) $\{6k : k \in N\}$

Sol. **(c)** We have,

$$(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$$

$$\Rightarrow (3 - 3\omega + 2\omega^2)^{4n+3} + \{\omega(3 - 3\omega + 2\omega^2)\}^{4n+3} + \{\omega^2(3 - 3\omega + 2\omega^2)\}^{4n+3} = 0$$

$$\Rightarrow (3 - 3\omega + 2\omega^2)^{4n+3}(1 + \omega^{4n+3} + \omega^{8n+6}) = 0$$

$$\Rightarrow (3 - 3\omega + 2\omega^2)^{4n+3}(1 + \omega^{4n} + \omega^{8n}) = 0$$

$$\Rightarrow (3 - 3\omega + 2\omega^2)^{4n+3}(1 + \omega^n + \omega^{2n}) = 0$$

$$\Rightarrow 1 + \omega^n + \omega^{2n} = 0$$

$\Rightarrow$ n is not a multiple of 3.

Hence, the set of values of n is $N - \{n : n = 3k, k \in N\}$

Smart Practice

1. If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the nth roots of unity and n is an odd natural number, then $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_{n-1})$ equals
(a) 1 *(b)* −1 *(c)* 0 *(d)* 2

2. If $\omega(\neq 1)$ be a cube root of unity and $(1+\omega^2)^n = (1+\omega^4)^n$, then the least positive value of n is [IIT JEE 2004]
(a) 2 *(b)* 3 *(c)* 5 *(d)* 6

3. If ω is a complex cube root of unity, then $\arg(i\omega) + \arg(i\omega^2)$ is equal to
(a) 0 *(b)* $\pi/2$
(c) π *(d)* None of these

4. If $z^2 + z + 1 = 0$, where z is a complex number, then the value of
$$\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \left(z^3+\frac{1}{z^3}\right)^2 + \ldots + \left(z^6+\frac{1}{z^6}\right)^2$$
is
(a) 54 *(b)* 6 *(c)* 12 *(d)* 18

5. If α is a non-real fifth root of unity, then the value of $3^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is
(a) 9 *(b)* 1 *(c)* 11/3 *(d)* 11

6. If z_1 and z_2 are two of the 8th roots of unity such that $\arg\left(\frac{z_1}{z_2}\right)$ is least positive, then $\frac{z_1}{z_2}$ is equal to
(a) $1+i$ *(b)* $1-i$
(c) $\frac{1+i}{\sqrt{2}}$ *(d)* $\frac{1-i}{\sqrt{2}}$

7. The number of roots of the equation $z^{15} = 1$ satisfying $|\arg(z)| < \frac{\pi}{2}$ is
(a) 6 *(b)* 7 *(c)* 8 *(d)* 9

8. Let ω be the cube root of unity. Then, the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is [IIT JEE 2002]
(a) 3ω *(b)* $3\omega(\omega-1)$
(c) $3\omega^2$ *(d)* $3\omega(1-\omega)$

9. If $\omega(\neq 1)$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$, then (A, B) equals [IIT JEE 2004]
(a) (0, 1) *(b)* (1, 1) *(c)* (1, 0) *(d)* (−1, 1)

10. If $\omega = e^{i2\pi/3}$ and a, b, c, x, y, z be non-zero complex numbers such that
$$a + b + c = x,$$
$$a + b\omega + c\omega^2 = y$$
and
$$a + b\omega^2 + c\omega = z.$$
Then, the value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is [IIT JEE 2011]
(a) 3 *(b)* 6 *(c)* 9 *(d)* 1

11. Let ω be the solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If a, b and c satisfying $[a\ b\ c]\begin{bmatrix} 1 & 9 & 7 \\ 2 & 8 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]$, then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to [IIT JEE 2011]
(a) −2 *(b)* 2 *(c)* 3 *(d)* −3

12. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$, then the value of $a_0 + a_3 + a_6 + \ldots$ is
(a) 1 *(b)* 2^n
(c) 2^{n-1} *(d)* 3^{n-1}

13. If ω is a complex cube root of unity, then a root of the equation
$$\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0 \text{ is}$$
(a) $x = 1$ *(b)* $x = \omega$
(c) $x = \omega^2$ *(d)* $x = 0$

14. If a, b, c are integers, not all equal and $\omega(\neq 1)$ is a cube root of unity, then minimum value of $|a + b\omega + c\omega^2|$ is
(a) $\sqrt{3}$ *(b)* 1 *(c)* 2 *(d)* 3

15. If $|z| = 3$, then the area of the triangle whose sides are $z, \omega z$ and $z + \omega z$ is (where, ω is a complex cube root of unity)
(a) $9\sqrt{3}/4$ *(b)* $3\sqrt{3}/2$
(c) 5/2 *(d)* $8\sqrt{3}/3$

16. If α, β are the roots of the equation $x^2 + px + q = 0$, ane ω is a cube root of unity, then the value of $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)$ is
(a) p^2 *(b)* $3q$
(c) $p^2 - 2q$ *(d)* $p^2 - 3q$

17. If $\omega(\neq 1)$ is a cube root of unity, then the value of $\tan\left[(\omega^{2017} + \omega^{2225})\pi - \frac{\pi}{3}\right]$ is
(a) $\frac{-1}{\sqrt{3}}$ *(b)* $\frac{1}{\sqrt{3}}$
(c) $-\sqrt{3}$ *(d)* $\sqrt{3}$

18. If α, β, γ are the cube roots of p, $(p < 0)$, then for any x, y and z which does not make denominator zero, the expression $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$ equals
(a) $\omega, 1$ *(b)* ω, ω^2 *(c)* $\omega^2, 1$ *(d)* $1, \omega, \omega^2$

19. If $\omega(\neq 1)$ is a cube root of unity and satisfies $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}=2\omega^2$ and $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}=2\omega$, then the value of $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$ is [IIT JEE 2004]

(a) 2 *(b)* –2
(c) $-1+\omega^2$ *(d)* None of these

20. If x^2-x+1 divides the polynomial $x^{n+1}-x^n+1$, then n must be of the form
(a) $3k+1$ *(b)* $6k+1$ *(c)* $6k-1$ *(d)* $3k-1$

21. Let $\omega(\neq 1)$ be a cube root of unity and $f: I \to C$ be defined by $f(n)=1+\omega^n+\omega^{2n}$, then the range of f is
(a) {0} *(b)* {0, 3} *(c)* {0, 1, 3} *(d)* {0, 1}

22. The value of $(i+\sqrt{3})^{100}+(i-\sqrt{3})^{100}+2^{100}$ is
(a) 0 *(b)* 1 *(c)* –1 *(d)* 2^{300}

23. If $x^2+x+1=0$, then the value of $\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\ldots+\left(x^{27}+\frac{1}{x^{27}}\right)^2$ is
(a) 27 *(b)* 72 *(c)* 45 *(d)* 54

24. If ω is a complex nth root of unity, then $\sum_{r=1}^{n}(ar+b)\omega^{r-1}$ is equal to
(a) $\frac{n(n+1)a}{2}$ *(b)* $\frac{nb}{1-n}$ *(c)* $\frac{na}{\omega-1}$ *(d)* nab

25. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x-1)^3+8=0$ are
(a) $-1, 1+2\omega, 1+2\omega^2$ *(b)* $-1, 1-2\omega, 1-2\omega^2$
(c) $-1, -1, -1$ *(d)* None of these

26. If $x=a+b$, $y=a\beta+b\gamma$, $z=a\gamma+b\beta$, where γ and β are complex cube roots of unity, then xyz is equal to
(a) a^3-b^3 *(b)* a^3+b^3 *(c)* abc *(d)* 0

27. If $x=\omega-\omega^2-2$, then the value of $x^4+3x^3+2x^2-11x-6$ is (where, ω is cube root of unity)
(a) –1 *(b)* 0
(c) 1 *(d)* 2

28. If $1, \omega, \omega^2$ are the cube roots of unity. The least possible degree of a polynomial with real coefficients having roots $2\omega, 2+3\omega, 2+3\omega^2, 2-\omega-\omega^2$ is
(a) 4 *(b)* 3
(c) 2 *(d)* 5

29. If $x+\frac{1}{x}=1$ and $p=x^{4000}+\frac{1}{x^{4000}}$ and q is the digit at unit place in the number $2^{2^n}+1$, then the value of $(p+q)$ is equal to
(a) 8 *(b)* 7 *(c)* 6 *(d)* 5

30. Let $z_k=\cos\left(\frac{2k\pi}{10}\right)+i\sin\left(\frac{2k\pi}{10}\right)$; $k=1, 2, \ldots, 9$. Then, $\frac{1}{10}\{|1-z_1||1-z_2|\ldots|1-z_9|\}$ equals to [JEE Advanced 2014]
(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

Answers with Explanation

1. *(a)* Since, $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the roots of $x^n-1=0$.

$\therefore \quad x^n-1=(x-1)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})$

$\Rightarrow \quad \frac{x^n-1}{x-1}=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})$

$\Rightarrow \quad (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_{n-1})$
$=x^{n-1}+x^{n-2}+\ldots+x^2+x+1 \quad \ldots(i)$

Putting $x=-1$ in the above equation, we get

$(-1-\alpha_1)(-1-\alpha_2)(-1-\alpha_3)\ldots(-1-\alpha_{n-1})$
$=(-1)^{n-1}+(-1)^{n-2}+\ldots+(-1)+(-1)^0$

$\Rightarrow \quad (-1)^{n-1}[(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots(1+\alpha_{n-1})]$
$=(-1)^{n-1}+(-1)^{n-2}+\ldots+(-1)+(-1)^0 \quad \ldots(ii)$

If n is odd, then RHS of the above equality contains an odd number of terms with alternately positive negative signs.

$\therefore \quad$ RHS = 1

Also, $\quad (-1)^{n-1}=1$

$\therefore \quad (1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots(1+\alpha_{n-1})=1$

2. *(b)* We have,

$(1+\omega^2)^n=(1+\omega^4)^n$

$\Rightarrow \quad \left(\frac{1+\omega^2}{1+\omega^4}\right)^n=1$

$\Rightarrow \quad \left(\frac{1+\omega^2}{1+\omega}\right)^n=1 \quad \Rightarrow \left(\frac{-\omega}{-\omega^2}\right)^n=1$

$\Rightarrow \quad \omega^{2n}=1 \quad \Rightarrow \quad n=3$

3. *(c)* We have, $\arg(\omega)=\frac{2\pi}{3}$ and $\arg(\omega^2)=\frac{4\pi}{3}$ or $-\frac{2\pi}{3}$

$\therefore \arg(i\omega)+\arg(i\omega^2)=\arg(i)+\arg(\omega)$
$+\arg(i)+\arg(\omega^2)$
$=\frac{\pi}{2}+\frac{2\pi}{3}+\frac{\pi}{2}-\frac{2\pi}{3}=\pi$

4. *(c)* We know that ω and ω^2 are roots of $z^2+z+1=0$ such that $\omega=\frac{1}{\omega^2}$. So, the value of the given expression remains same for $z=\omega$ and $z=\frac{1}{\omega^2}$. Also, $1+\omega+\omega^2=0$ and $\omega^3=1$.

$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\ldots+\left(z^6+\frac{1}{z^6}\right)^2$

$$= \left\{\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^4 + \frac{1}{z^4}\right)^2 + \left(z^5 + \frac{1}{z^5}\right)^2\right\}$$
$$+ \left\{\left(z^3 + \frac{1}{z^3}\right)^2 + \left(z^6 + \frac{1}{z^6}\right)^2\right\}$$
$$= 4(\omega + \omega^2)^2 + (2^2 + 2^2)$$
$$= 4(-1)^2 + 4 + 4$$
$$= 4 + 4 + 4 = 12$$

5. *(a)* We have,
$$\alpha = 1^{1/5}, \alpha \neq 1$$
$$\Rightarrow \quad \alpha^5 = 1$$
$$\Rightarrow \quad \alpha^{-2} = \alpha^3 \text{ and } \alpha^{-1} = \alpha^4$$
$$\therefore \quad 1 + \alpha + \alpha^2 + \alpha^{-2} - \alpha^{-1} = 1 + \alpha + \alpha^2 + \alpha^3 - \alpha^4$$
$$= \frac{\alpha^4 - 1}{\alpha - 1} - \alpha^4 = \frac{\frac{1}{\alpha} - 1}{\alpha - 1} - \frac{1}{\alpha} \quad \left[\because \alpha^5 = 1 \Rightarrow \alpha^4 = \frac{1}{\alpha}\right]$$
$$= -\frac{1}{\alpha} - \frac{1}{\alpha} = -\frac{2}{\alpha} = -2\alpha^4$$
Now, $|1 + \alpha + \alpha^2 + \alpha^{-2} - \alpha^{-1}| = |-2\alpha^4| = 2|\alpha|^4 = 2$
$$\therefore \quad 3^{|1 + \alpha + \alpha^2 + \alpha^{-2} - \alpha^{-1}|} = 3^2 = 9$$

6. *(c)* We know that nth root of unity lie on the circle $|z| = 1$ and divide its circumference into n equal parts.
Also, nth roots of unity are given by
$$z_r = e^{\frac{i2r\pi}{n}}; r = 0, 1, 2, \ldots, (n-1)$$
$\therefore$ 8th roots of unity are given by
$$z_r = e^{\frac{ir\pi}{4}}; r = 0, 1, 2, \ldots, 7$$
We know that, $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
Therefore, $\arg\left(\frac{z_1}{z_2}\right)$ will have least positive value, if z_1 and z_2 represent any two consecutive points on the circle $|z| = 1$.
Let $\quad z_1 = z_r = e^{\frac{ir\pi}{4}}$
Thus, $\quad z_2 = z_{r-1} = e^{\frac{i(r-1)\pi}{4}}$
$$\therefore \quad \frac{z_1}{z_2} = e^{\frac{i\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$
$$= \frac{1 + i}{\sqrt{2}}$$

7. *(b)* We have, $z^{15} = 1 \Rightarrow z = (1)^{\frac{1}{15}}$
$$\Rightarrow \quad z = e^{\frac{i2\pi r}{15}}; r = 0, 1, 2, \ldots, 14$$
$$\Rightarrow \quad z = e^{\pm\frac{2\pi r}{15}}; r = 0, 1, 2, \ldots, 7$$
Thus, the complex numbers having their arguments between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ are
$$(e^0 = 1), e^{\pm\frac{i2\pi}{15}}, e^{\pm\frac{i4\pi}{15}}, e^{\pm\frac{i6\pi}{15}}$$
Clearly, the total number of these complex numbers is 7.

8. *(b)* We have,
$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2-\omega^2 & \omega^2 - 1 \\ 1 & \omega^2 - 1 & \omega - 1 \end{vmatrix}$$
$$[\text{applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$$
$$= (-2 - \omega^2)(\omega - 1) - (\omega^2 - 1)^2$$
$$= -2\omega + 2 - \omega^3 + \omega^2 - \omega^4 + 2\omega^2 - 1$$
$$= -2\omega + 2 - 1 + \omega^2 - \omega + 2\omega^2 - 1$$
$$= -3\omega + 3\omega^2 = 3\omega(\omega - 1)$$

9. *(b)* We have,
$$(1 + \omega)^7 = A + B\omega$$
$$\Rightarrow \quad (-\omega^2)^7 = A + B\omega$$
$$\Rightarrow \quad -\omega^{14} = A + B\omega$$
$$\Rightarrow \quad -\omega^2 = A + B\omega \qquad [\because \omega^{3n+r} = \omega^r]$$
$$\Rightarrow \quad 1 + \omega = A + B\omega \qquad [\because 1 + \omega + \omega^2 = 0]$$
$$\Rightarrow \quad A = 1, B = 1$$

10. *(a)* We have,
$$x = a + b + c$$
$$\Rightarrow \quad |x|^2 = x\bar{x} = (a + b + c)(\overline{a + b + c})$$
$$|x|^2 = a\bar{a} + b\bar{b} + c\bar{c} + (a\bar{b} + \bar{a}b) + (b\bar{c} + \bar{b}c) + (c\bar{a} + \bar{c}a)$$
and $\quad y = a + b\omega + c\omega^2$
$$\Rightarrow \quad |y|^2 = y\bar{y} = (a + b\omega + c\omega^2)(\overline{a + b\omega + c\omega^2})$$
$$= (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\omega^2 + \bar{c}\omega)$$
$$\Rightarrow \quad |y|^2 = a\bar{a} + b\bar{b} + c\bar{c} + (a\bar{b}\omega^2 + \bar{a}b\omega) + (b\bar{c}\omega^2 + \bar{b}c\omega) + (\bar{c}a\omega + c\bar{a}\omega^2)$$
$$\Rightarrow \quad |y|^2 = |a|^2 + |b|^2 + |c|^2 + (a\bar{b}\omega^2 + \bar{a}b\omega) + (b\bar{c}\omega^2 + \bar{b}c\omega) + (\bar{c}a\omega + c\bar{a}\omega^2)$$
Similarly,
$$|z|^2 = |a|^2 + |b|^2 + |c|^2 + (a\bar{b}\omega + \bar{a}b\omega^2) + (b\bar{c}\omega + \bar{b}c\omega^2) + (\bar{c}a\omega^2 + c\bar{a}\omega)$$
$$\therefore \quad |x|^2 + |y|^2 + |z|^2 = 3\{|a|^2 + |b|^2 + |c|^2\}$$
$$\Rightarrow \quad \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$$

11. *(a)* We have,
$$[a\ b\ c]\begin{bmatrix} 1 & 9 & 7 \\ 2 & 8 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]$$
$$\Rightarrow \quad [a + 2b + 7c \quad 9a + 8b + 3c \quad 7a + 7b + 7c] = [0\ 0\ 0]$$
$$\Rightarrow \quad a + 2b + 7c = 0, 9a + 8b + 3c = 0, 7a + 7b + 7c = 0$$
$$\Rightarrow \quad a = 1, b = 6, c = -7$$
Clearly, $\omega = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$ and it satisfies $\omega^3 = 1$.
$$\therefore \quad \frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega} + \frac{1}{\omega^6} + \frac{3}{\omega^{-7}}$$
$$= 3\omega^2 + 1 + 3\omega = 3(\omega^2 + \omega) + 1$$
$$= -3 + 1 = -2$$

12. *(d)* Putting $x = 1, \omega, \omega^2$ in
$(1 + x + x^2)^n = (a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n})$, we get
$(1 + 1 + 1)^n = a_0 + a_1 + a_2 + a_3 + \ldots + a_{2n}$,

$(1+\omega+\omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \ldots + a_{2n}\omega^{2n}$
and $(1+\omega^2+\omega^4)^n = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \ldots + a_{2n}\omega^{4n}$
Adding the above three equations and using $1+\omega+\omega^2 = 0, \omega^3 = 1$, we get

$$3^n = 3(a_0 + a_3 + a_6 + \ldots)$$

$$\Rightarrow \quad a_0 + a_3 + a_6 + \ldots = 3^{n-1}$$

13. *(d)* We have,

$$\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+1+\omega+\omega^2 & \omega & \omega^2 \\ x+1+\omega+\omega^2 & x+\omega^2 & 1 \\ x+1+\omega+\omega^2 & 1 & x+\omega \end{vmatrix} = 0$$

[applying $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow \begin{vmatrix} x & \omega & \omega^2 \\ x & x+\omega^2 & 1 \\ x & 1 & x+\omega \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & \omega-1 & \omega^2-x-\omega \\ 0 & x+\omega^2-1 & 1-x-\omega \\ x & 1 & x+\omega \end{vmatrix} = 0$$

[applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$]

$$\Rightarrow \quad x[(\omega-1)(1-x-\omega) - (x+\omega^2-1)(\omega^2-x-\omega)] = 0$$

$$\Rightarrow \quad x[\omega - x\omega - \omega^2 - 1 + x + \omega - x\omega^2 + x^2 + x\omega - \omega^4 + x\omega^2 + \omega^3 + \omega^2 + x - \omega] = 0$$

$$\Rightarrow \quad x(x^2 + 2x) = 0$$

$$\Rightarrow \quad x^2(x+2) = 0$$

$$\Rightarrow \quad x = 0, -2$$

14. *(b)* We know that,

$$|a+b\omega+c\omega^2|^2 = (a+b\omega+c\omega^2)\,\overline{(a+b\omega+c\omega^2)}$$
$$= (a+b\omega+c\omega^2)(a+b\bar{\omega}+c\bar{\omega}^2)$$
$$= (a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$
$$= a^2+b^2+c^2+(bc+ca+ab)(\omega+\omega^2)$$
$$= a^2+b^2+c^2-bc-ca-ab$$
$$= \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$$

As a, b, c are integers not all equal atleast two of $(b-c)$, $(c-a)$, $(a-b)$ are of different from zero. Therefore, minimum value of $(b-c)^2+(c-a)^2+(a-b)^2$ is 2.

Thus, $\quad |a+b\omega+c\omega^2|^2 \geq 1$

$$\Rightarrow \quad |a+b\omega+c\omega^2| \geq 1$$

15. *(a)* We have,

$$|z| = 3$$

$$\Rightarrow \quad |\omega z| = |\omega||z| = (1)(3) = 3$$

Also, $\quad |z+\omega z| = |z(1+\omega)|$
$$= |z(-\omega)^2|$$
$$= |z||-\omega^2|$$
$$= (3)(1) = 3$$

$\therefore$ Given triangle is equilateral triangle and its area

$$= \frac{\sqrt{3}}{4}|z|^2 = \frac{\sqrt{3}}{4} \times (3)^2 = \frac{9\sqrt{3}}{4}$$

16. *(d)* We have, $\alpha+\beta = -p, \alpha\beta = q$

Now, $(\omega\alpha+\omega^2\beta)(\omega^2\alpha+\omega\beta)$
$$= \omega^3\alpha^2 + \omega^4\alpha\beta + \omega^2\alpha\beta + \omega^3\beta^2$$
$$= \alpha^2 + \beta^2 + (\omega+\omega^2)\alpha\beta$$
$$= \alpha^2 + \beta^2 - \alpha\beta$$
$$= (\alpha+\beta)^2 - 3\alpha\beta$$
$$= p^2 - 3q$$

17. *(c)* We have,

$$\omega^{2017} = (\omega^3)^{672}\omega = \omega$$

and $\quad \omega^{2225} = (\omega^3)^{741}\omega^2 = \omega^2$

$$\therefore \tan\left[(\omega^{2017} + \omega^{2225})\pi - \frac{\pi}{3}\right]$$
$$= \tan\left[(\omega+\omega^2)\pi - \frac{\pi}{3}\right]$$
$$= \tan\left[-\pi - \frac{\pi}{3}\right] = -\tan\left[\pi + \frac{\pi}{3}\right]$$
$$= -\tan\frac{\pi}{3} = -\sqrt{3}$$

18. *(b)* We have,

$$x^3 = p \Rightarrow x = p^{1/3}$$

$$\Rightarrow \quad x = p^{1/3}, p^{1/3}\omega, p^{1/3}\omega^2$$

Let $\quad \alpha = p^{1/3}, \beta = p^{1/3}\omega, \gamma = p^{1/3}\omega^2$

Now, $\quad \dfrac{x\alpha+y\beta+z\gamma}{x\beta+y\gamma+z\alpha} = \dfrac{x+y\omega+\omega^2 z}{x\omega+y\omega^2+z}$

$$= \frac{1}{\omega} = \omega^2$$

If $\alpha = p^{1/3}\omega, \beta = p^{1/3}, \gamma = p^{1/3}\omega^2$, then

$$\frac{x\alpha+y\beta+z\gamma}{x\beta+y\gamma+z\alpha} = \frac{x\omega+y+z\omega^2}{x+y\omega^2+z\omega} = \frac{1}{\omega^2} = \omega$$

19. *(a)* Note that ω, ω^2 are roots of

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} = \frac{2}{x}$$

$$\Rightarrow x[bc+ca+ab+2(a+b+c)x+3x^2]$$
$$= 2[abc+(bc+ca+ab)x+(a+b+c)x^2+x^3]$$

$$\Rightarrow \quad x^3 - (bc+ca+ab)x - 2abc = 0$$

If α is the third root of this equation, then

$$\alpha+\omega+\omega^2 = 0 \Rightarrow \alpha = 1$$

$$\therefore \quad \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = \frac{2}{1} = 2$$

20. *(b)* We know that,

$$x^2 - x + 1 = (x+\omega)(x+\omega^2)$$

Therefore, $-\omega, -\omega^2$ must be zeroes of

$$x^{n+1} - x^n + 1.$$

Now, $(-\omega)^{n+1} - (\omega)^n + 1$
$$= (-1)^{n+1}\omega^n(\omega+1) + 1$$
$$= (-1)^{n+1}\omega^n(-\omega^2) + 1$$
$$= (-1)^{n+2}\omega^{n+2} + 1 = 0, \text{ if } n = 6k+1$$

21. *(b)* If $n = 3k, k \in I$, then $f(n) = 1+1+1 = 3$

If $n = 3k+1, k \in I$, then $f(n) = 1+\omega+\omega^2 = 0$

If $n = 3k+2, k \in I$, then $f(n) = 1+\omega^2+\omega = 0$

Thus, the range of f is $\{0, 3\}$.

22. *(a)* We have,

$$(i+\sqrt{3})^{100}+(i-\sqrt{3})^{100}+2^{100}$$

$$=\left(\frac{i^2+i\sqrt{3}}{i}\right)^{100}+\left(\frac{i^2-i\sqrt{3}}{i}\right)^{100}+2^{100}$$

$$=\frac{2^{100}}{i^{100}}\left(\frac{-1+i\sqrt{3}}{2}\right)^{100}+\frac{2^{100}}{i^{100}}\left(\frac{-1-i\sqrt{3}}{2}\right)^{100}+2^{100}$$

$$=2^{100}(\omega)^{100}+2^{100}(\omega^2)^{100}+2^{100}$$

$$=2^{100}(\omega^{100}+\omega^{200}+1)$$

$$=2^{100}(\omega+\omega^2+1)$$

$$=2^{100}\times 0=0$$

23. *(d)* Given, $x^2+x+1=0$

$\Rightarrow \quad x=\omega$ or ω^2

Let $\quad x=\omega$

Then, $x+\frac{1}{x}=\omega+\frac{1}{\omega}=\omega+\omega^2=-1$

$$x^2+\frac{1}{x^2}=\omega^2+\frac{1}{\omega^2}=\omega^2+\omega=-1$$

$$x^3+\frac{1}{x^3}=\omega^3+\frac{1}{\omega^3}=2$$

$$x^4+\frac{1}{x^4}=\omega^4+\frac{1}{\omega^4}$$

$$=\omega+\omega^2=-1, \text{ etc.}$$

$$\therefore\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\ldots+\left(x^{27}+\frac{1}{x^{27}}\right)^2$$

$$=\left[\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^4+\frac{1}{x^4}\right)^2+\ldots+\left(x^{26}+\frac{1}{x^{26}}\right)^2\right]$$

$$+\left[\left(x^3+\frac{1}{x^3}\right)^2+\left(x^6+\frac{1}{x^6}\right)^2+\left(x^9+\frac{1}{x^9}\right)^2+\ldots+\left(x^{27}+\frac{1}{x^{27}}\right)^2\right]$$

$$=18+9(2)^2=54$$

24. *(c)* $E=\sum_{r=1}^{n}(ar+b)\omega^{r-1}$

$$=(a+b)+(2a+b)\omega+(3a+b)\omega^2+\ldots+(na+b)\omega^{n-1}$$

$$=b\underbrace{(1+\omega+\omega^2+\ldots+\omega^{n-1})}_{\text{zero}}+a(1+2\omega+3\omega^2+\ldots+n\omega^{n-1})$$

Now, $\quad S=1+2\omega+3\omega^2+\ldots+n\omega^{n-1}$

$$S\omega=\quad \omega+2\omega^2+\ldots+(n-1)\omega^{n-1}+n\omega^n$$

$$\therefore \quad S(1-\omega)=\underbrace{1+\omega+\omega^2+\ldots+\omega^{n-1}}_{\text{zero}}-n\omega^n$$

$$=-n \qquad [\because \omega^n=1]$$

$$\Rightarrow \quad S=\frac{n}{\omega-1}$$

$$\Rightarrow \quad E=\frac{na}{\omega-1}$$

25. *(b)* We have,

$$(x-1)^3+8=0$$

$$\Rightarrow \quad (x-1)^3=-8 \;\Rightarrow\; \left(\frac{x-1}{-2}\right)^3=1$$

$$\Rightarrow \quad \frac{x-1}{-2}=1,\omega,\omega^2 \Rightarrow x=-1,1-2\omega,1-2\omega^2$$

26. *(b)* As β and γ are the complex cube roots of unity, therefore let $\beta=\omega$ and $\gamma=\omega^2$, so that $\omega+\omega^2+1=0$ and $\omega^3=1$.

Then, $\quad xyz=(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)$

$$=(a+b)(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3)$$

$$=(a+b)(a^2+ab\omega+ab\omega^2+b^2)$$

$$=(a+b)[a^2+ab(\omega+\omega^2)+b^2]$$

$$=(a+b)(a^2-ab+b^2)=a^3+b^3$$

27. *(c)* We have,

$$x=\omega-\omega^2-2$$

$$\Rightarrow \quad x+2=\omega-\omega^2$$

$$\Rightarrow \quad (x+2)^2=(\omega-\omega^2)^2$$

$$\Rightarrow \quad x^2+4x+4=\omega^2+\omega^4-2\omega^3$$

$$\Rightarrow \quad x^2+4x+4=\omega^2+\omega-2$$

$$\Rightarrow \quad x^2+4x+4=-3$$

$$\Rightarrow \quad x^2+4x+7=0$$

Now, $x^4+3x^3+2x^2-11x-6$

$$=(x^2+4x+7)(x^2-x-1)+1$$

$$=(0)(x^2-x-1)+1=1$$

28. *(d)* Roots are $2\omega,(2+3\omega),(2+3\omega^2),(2-\omega-\omega^2),(2+3\omega)$ and $(2+3\omega^2)$ are conjugate to each other.

2ω is complex root, then other root must be $2\omega^2$.

$2-\omega-\omega^2=2-(-1)=3$, which is real.

Hence, least degree of the polynomial is 5.

29. *(c)* We have,

$$x+\frac{1}{x}=1 \quad \Rightarrow x^2-x+1=0$$

$\therefore \quad x=-\omega,-\omega^2$

Now, for $\quad x=-\omega,$

$$p=\omega^{4000}+\frac{1}{\omega^{4000}}=\omega+\frac{1}{\omega}=-1$$

Similarly, for $x=-\omega^2$, $p=-1$ and for $n>1$,

$$2^n=4k$$

$$\therefore \quad 2^{2^n}=2^{4k}$$

$$=(16)^k=\text{A number with last digit 6.}$$

$$\Rightarrow \quad q=6+1=7$$

Hence, $\quad p+q=-1+7=6$

30. *(b)* Clearly, $1, z_1, z_2, \ldots, z_9$ are 10th roots of unity.

$$\therefore z^{10}-1=(z-1)(z-z_1)(z-z_2)\ldots(z-z_9)$$

$$\Rightarrow \quad (z-z_1)(z-z_2)\ldots(z-z_9)=\frac{z^{10}-1}{z-1}$$

$$\Rightarrow \quad (z-z_1)(z-z_2)\ldots(z-z_9)=1+z+z^2+\ldots+z^9$$

$$\Rightarrow \quad (1-z_1)(1-z_2)\ldots(1-z_9)=10$$

$$\Rightarrow \quad \frac{1}{10}\{|1-z_1||1-z_2|\ldots|1-z_9|\}=1$$

CALCULUS IN THE REAL WORLD

Although there are many uses of calculus in real world. One of the most common use of calculus in our daily life is the PID controller. PID stands for Proportional Integral Derivative Controller.

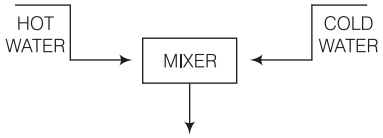
We have tried to explain with very simple analogies for the layman. It has a wide range of applications :

1. Coffee Machines
2. Automatic Air Conditioners
3. Cruise Control in Cars
4. Water Mixers
5. Online Advertisements
6. Industrial control system, Ships, Missiles, Rockets, UAV etc.

PID is a kind of continuous machine learning loop where you need to control the flow of the output without having any prior information of the input.

Water Mixer Analogy

Consider a mixing instrument which is supposed to mix hot and cold water in the right proportion, so that the mixtures flows out at exactly 35°C.



This is very easy if you know the exact temperature of the input water flow and other constants like the thickness of pipe.

If you don't know the input temperatures or if the input temperature of hot and cold waters are variable and keep changing every hour or every minute, the problem gets complex. Industrial applications face even more volatility.

The PID controller opens the hot valve slightly to measure the change in the output temperature. It will then keep changing it frequently by a constant value. If the desired temperature is far away, it automatically increases the rate of change. So, it will first open at x, then $2x$, $10x$ etc. till it overshoots the desired temperature and say reaches 50°. It will then turn the process back and decrease it till it reaches 35. In case it overshoots again to 30, it goes up again oscillating around the target temperature till the error goes infinitesimally small.

Online Advertising Analogy

You launch an ad campaign on facebook or google. You target all the Indian users in the age group 18-22, who are females. You have a spend budget of ₹ 20,000 per day. There are 100 other advertisers with similar complex targetting criteria with different budgets. Whenever, you load a page, the delivery system is supposed to make a decision on which ad to serve. It is at the same time supposed to make sure that budgets of all clients are spent evenly throughout the day. The usage pattern of users throughout the day is unpredictable. It becomes a very complex problem if you consider all the dynamics involved.

The controller in the ad delivery system will learn and set a pacing value for every campaign. This is the rate at which that campaign is supposed to be delivered. The pacing is set at a certain value at midnight. Post that it will be adjusted every minute by the PID algorithm, so that the delivery for all campaign remain smooth. If you change your campaign's budget in the afternoon, it will automatically change the delivery pacing to meet the goal.

Now, can you think of more things that might be applications of calculus.

BINOMIAL THEOREM

A collection of chapterwise Statement type problems for JEE Main

Directions (Q. Nos. 1-30) In these questions, Statement I is followed by a corresponding Statement II. Use the following options to choose the appropriate answer.

(a) Both the Statements are true and Statement II is the correct explanation of Statement I

(b) Both the Statements are true but Statement II is not the correct explanation of Statement I

(c) Statement I is true but Statement II is false

(d) Statement I is false but Statement II is true

1. **Statement I** $\sum_{r=0}^{n} r^2\, {}^nC_r x^r = n(n-1)x^2(1+x)^{n-2} + nx(1+x)^{n-1}$

Statement II $\sum_{r=0}^{n} r^2\, {}^nC_r = n(n-1)\,2^{n-2} + n\,2^{n-1}$

2. Let a, b, c be the sides of a ΔABC opposite to angles A, B, C, respectively.

Let $\alpha = \sum_{r=0}^{n} {}^nC_r b^{n-r} c^r \cos\{rB-(n-r)C\}$

and $\beta = \sum_{r=0}^{n} {}^nC_r b^{n-r} c^r \sin\{rB-(n-r)C\}$.

Statement I $\alpha = a^n$

Statement II $\beta = a^n$

3. Let $(1+x)^{36} = a_0 + a_1x + a_2x^2 + \ldots + a_{36}x^{36}$, then

Statement I $a_0 + a_3 + a_6 + \ldots + a_{36} = \frac{2}{3}(2^{35}+1)$

Statement II $a_0 + a_2 + a_4 + \ldots + a_{36} = 2^{35}$

4. **Statement I** $\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{r+1} = \frac{1}{n+1}$

Statement II $\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{r+1} x^r = \frac{1}{(n+1)x}\{1-(1-x)^{n+1}\}$

5. Let $S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$; $S_2 = \sum_{j=1}^{10} j\cdot {}^{10}C_j$

and $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j$.

Statement I $S_3 = 50\times 2^9$

Statement II $S_1 = 90\times 2^8$ and $S_2 = 10\times 2^8$

6. **Statement I** Coefficient of x^n in $\sum_{j=0}^{n} (-1)^j (2x+3)^{n-j}(5-2x)^j$ is $(n+1)\,2^n$.

Statement II $\sum_{j=0}^{n} a^{n-j}\cdot b^j = \frac{a^{n+1}-b^{n+1}}{a-b}$, $a \neq b$ and coefficient of x^m in $(\alpha x+\beta)^{m+1}$ is $(m+1)\,\alpha^m\beta$.

7. Let $(1+x)^{3n} = C_0 + C_1x + C_2x^2 + \ldots + C_{3n}x^{3n}$ and $\omega\,(\neq 1)$ be a cube root of unity.

Statement I $C_0 + C_1\omega + C_2\omega^2 + C_3 + C_4\omega + C_5\omega^2 + \ldots = (-1)^n$

Statement II Cube root unity from a triangle of area $\sqrt{3}$ sq units.

8. Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Statement I $5C_0^2 + 7C_1^2 + 9C_2^2 + \ldots + (5+2n)C_n^2 = (5+n)\frac{(2n)!}{n!\,n!}$

Statement II $C_0^2 + C_1^2 + \ldots + C_n^2 = {}^{2n}C_n$

9. Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Statement I For $m \geq 2$, $C_0 - C_1 + C_2 - \ldots + (-1)^{m-1} C_{m-1} = (-1)^{m-1}({}^{n-1}C_{m-1})$

Statement II ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ for $1 \leq r \leq n-1$

10. Let $(1+t)^n = C_0 + C_1t + C_2t^2 + \ldots + C_nt^n$

Statement I $\frac{C_0}{(1)(2)} + \frac{C_1}{(2)(3)} + \frac{C_2}{(3)(4)} + \ldots + \frac{C_n}{(n+1)(n+2)} = \frac{1}{n+1}\left[\frac{2^{n+2}}{n+2} - 1\right]$

Statement II $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \ldots + (-1)^n \frac{C_n}{n+2} = 0$

11. **Statement I** Greatest term in the expansion of $(\sqrt{3}+\sqrt{2})^{50}$ is $\binom{50}{22}3^{14}2^{11}$.

Statement II Greatest term in the expansion of $(1+x)^n$, $x>0$ is the rth term, if $\frac{(n+1)x}{x+1}$ is not an integer and $r=\left[\frac{(n+1)x}{x+1}\right]$, where $[y]$ denotes the greatest integer $\le y$.

12. **Statement I** For any positive integers m, n (with $n \ge m$),

$$\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{n}=\binom{n+1}{m}$$

Statement II Coefficient of x^r in the expansion of $(1+x)^n$ is $\binom{n}{r}$.

13. Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Statement I $\sum_{r=0}^{n} C_r \sin(rx)\cos(n-r)x = 2^n \sin(nx)$

Statement II $\sum_{r=0}^{n} C_r = 2^n$

14. **Statement I** The value of

${}^{10}C_0 + ({}^{10}C_0 + {}^{10}C_1) + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2)$

$+ \ldots + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_9)$ is $10 \ldots 2^9$.

Statement II

${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots + n\,{}^nC_n = n \cdot 2^{n-1}$

15. **Statement I** Remainder when 3456^{2222} is divided by 7 is 4.

Statement II Remainder when 5^{2222} is divided by 7 is 4.

16. **Statement I** If $n \in N$ and n is not a multiple of 3 and $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then the value of $\sum_{r=0}^{n} (-1)^r a_r\,{}^nC_r$ is zero.

Statement II The coefficient of x^n in the expansion of $(1-x^3)$ is zero, if $n = 3k+1$ as $n = 3k+2$.

17. **Statement I** Greatest term in the expansion of $(1+x)^{12}$, when $x=\frac{11}{10}$, is 7th.

Statement II 7th term in the expansion of $(1+x)^{12}$ has the factor ${}^{12}C_6$ which is the greatest value of ${}^{12}C_r$.

18. **Statement I** Three consecutive binomial coefficients are always in AP.

Statement II Three consecutive binomial coefficients are not in HP or GP.

19. **Statement I** The number of distinct terms in $(1+x+x^2+x^3+x^4)^{1000}$ is 4001.

Statement II The number of distinct terms in the expansion $(a_1+a_2+\ldots+a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

20. **Statement I** The total number of dissimilar terms in the expansion of $(x_1+x_2+\ldots+x_n)^3$ is $\frac{n(n+1)(n+2)}{6}$.

Statement II The total number of dissimilar terms in the expansion of $(x_1+x_2+x_3)^n$ is $\frac{n(n+1)(n+2)}{6}$.

21. **Statement I** $3^{2n+2} - 8n - 9$ is divisible by 64, $\forall\, n \in N$.

Statement II $(1+x)^n - nx - 1$ is divisible by x^2, $\forall n \in N$.

22. **Statement I** The coefficient of x^n in $\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}\right)$ is $\frac{3^n}{n!}$.

Statement II The coefficient of x^n in e^{3x} is $\frac{3^n}{n!}$.

23. **Statement I** In the expansion of $(1+x)^{41}(1-x+x^2)^{40}$, the coefficient of x^{85} is zero.

Statement II In the expansion of $(1+x)^{41}$ and $(1-x+x^2)$, x^{85} terms does not occur.

24. **Statement I** $\sum_{0\le i<j\le n}\left(\frac{i}{{}^nC_i}+\frac{j}{{}^nC_j}\right)$ is equal to $\frac{n^2}{2}a$, where $a=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$.

Statement II $\sum_{r=0}^{n}\frac{r}{{}^nC_r}=\sum_{r=0}^{n}\frac{n-r}{{}^nC_r}$

25. **Statement I** If P is a prime number ($P \ne 2$), then $[(2+\sqrt{5})^P] - 2^{P+1}$ is always divisible by P (where, $[\cdot]$ denotes the greatest integer function).

Statement II If n is prime, then ${}^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_{n-1}$ must be divisible by n.

26. **Statement I** ${}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^nC_r = 0$, if $m+n<r$.

Statement II ${}^nC_r = 0$, if $n<r$.

27. **Statement I** The sum of coefficients in the expansion of $(3^{-x/4}+3^{5x/4})^n$ is 2^n.

Statement II The sum of coefficients in the expansion of $(x+y)^n$ is 2^n when we put $x=y=1$.

28. Let n be a positive integer and k be a whole number $k \le 2n$.

Statement I The maximum value of ${}^{2n}C_k$ is ${}^{2n}C_n$.

Statement II $\frac{{}^{2n}C_{k+1}}{{}^{2n}C_k} < 1$, for $k = 0, 1, 2, \ldots, n-1$ and $\frac{{}^{2n}C_k}{{}^{2n}C_{k-1}} > 1$, for $k = n+1, n+2, \ldots, 2n$

29. **Statement I** The coefficient of the term of independent of x in the expansion of $\left(x + \frac{9}{x} + 6\right)^n$ is $\frac{3^n(2n)!}{n!\,n!}$.

Statement II The coefficient of x^r in the expansion of $(1+x)^n$ is $\binom{n}{r}$.

30. **Statement I** For each natural number n, $(n+1)^7 - n^7 - 1$ is divisible by 7.

Statement II For each natural number n, $n^7 - n$ is divisible by 7.

Answers with Explanation

1. ***(b)*** We have,

$$\sum_{r=0}^{n} r^2\,{}^nC_r x^r$$
$$= \sum_{r=0}^{n} \{r(r-1)+r\}\,{}^nC_r x^r$$
$$= \sum_{r=0}^{n} r(r-1)\,{}^nC_r x^r + \sum_{r=0}^{n} r\,{}^nC_r x^r$$
$$= \sum_{r=2}^{n} r(r-1)\times\frac{n}{r}\times\frac{n-1}{r-1}\,{}^{n-2}C_{r-2}\,x^{r-2}x^2 + \sum_{r=1}^{n} r\times\frac{n}{r}\times{}^{n-1}C_{r-1}\,x^{r-1}\times x$$
$$= n(n-1)x^2\sum_{r=2}^{n}{}^{n-2}C_{r-2}\,x^{r-2} + nx\sum_{r=1}^{n}{}^{n-1}C_{r-1}\,x^{r-1}$$
$$= n(n-1)x^2(1+x)^{n-2} + nx(1+x)^{n-1}$$

So, Statement I is true.

Replacing x by 1 is Statement I, we get

$$\sum_{r=0}^{n} r^2\,{}^nC_r = n(n-1)\,2^{n-2} + n2^{n-1}$$

So, Statement II is also true.

Clearly, Statement I is a correct explanation for Statement II. But Statement II is not a correct explanation for Statement I.

2. ***(c)*** We have,

$$\alpha + i\beta = \sum_{r=0}^{n}{}^nC_r b^{n-r}c^r e^{i\{rB-(n-r)c\}}$$
$$\Rightarrow\quad \alpha + i\beta = \sum_{r=0}^{n}{}^nC_r(be^{-ic})^{n-r}(ce^{iB})^r$$
$$\Rightarrow\quad \alpha + i\beta = (be^{-ic} + ce^{iB})^n$$
$$\Rightarrow\quad \alpha + i\beta = \{(b\cos C + c\cos B) + i(-b\sin C + c\sin B)\}^n$$
$$\Rightarrow\quad \alpha + i\beta = (a + i0)^n \qquad \left[\because a = b\cos C + c\cos B \text{ and } \frac{b}{\sin B} = \frac{c}{\sin C}\right]$$
$$\Rightarrow\quad \alpha + i\beta = a^n$$
$$\Rightarrow\quad \alpha = a^n \text{ and } \beta = 0$$

3. ***(b)*** We have,

$$a_0 + a_1 + a_2x^2 + a_3x^3 + \ldots + a_{36}x^{36} = (1+x)^{36} \quad \ldots(\text{i})$$

Putting $x = 1, \omega, \omega^2$ successively, we get

$$(a_0 + a_3 + a_6 + \ldots + a_{36}) + (a_1 + a_4 + a_7 + \ldots + a_{34}) + (a_2 + a_5 + \ldots + a_{35}) = 2^{36} \quad \ldots(\text{ii})$$
$$(a_0 + a_3 + a_6 + \ldots + a_{36}) + \omega(a_1 + a_4 + a_7 + a_{34}) + \omega^2(a_2 + a_5 + \ldots + a_{35}) = 1 \quad \ldots(\text{iii})$$
$$(a_0 + a_3 + a_6 + \ldots + a_{36}) + \omega^2(a_1 + a_4 + a_7 + \ldots + a_{34}) + \omega(a_2 + a_5 + \ldots + a_{35}) = 1 \quad \ldots(\text{iv})$$

Adding these three equations, we get

$$3(a_0 + a_3 + a_6 + \ldots + a_{36}) = 2(2^{35} + 1)$$
$$\Rightarrow\quad a_0 + a_3 + a_6 + \ldots + a_{36} = \frac{2}{3}(2^{35} + 1)$$

So, Statement I is true.

Putting $x = -1$ in Eq. (i) and adding it to Eq. (ii), we get

$$a_0 + a_2 + a_4 + \ldots + a_{36} = 2^{35}$$

So, Statement II is true. But, Statement II is not a correct explanation for Statement I.

4. ***(a)*** We have, $\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{r+1}x^r$

$$= -\frac{1}{x(n+1)}\sum_{r=0}^{n}(-1)^{r+1}\frac{n+1}{r+1}\,{}^nC_r\,x^{r+1}$$
$$= -\frac{1}{x(n+1)}\sum_{r=0}^{n}(-1)^{r+1}\,{}^{n+1}C_{r+1}\,x^{r+1}$$
$$= \frac{-1}{x(n+1)}\{(1-x)^{n+1} - 1\}$$
$$= \frac{1}{(n+1)x}\{1-(1-x)^{n+1}\}$$

So, Statement II is true.

Replacing x by 1 in Statement II, we get

$$\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{r+1} = \frac{1}{n+1}$$

So, Statement I is also true and Statement II is a correct explanation for Statement I.

5. ***(d)*** We have,

$$S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j = \sum_{j=1}^{10} j\times\frac{10}{j}\,{}^9C_{j-1}$$

$$= 10\sum_{j=1}^{10} {}^9C_{j-1} = 10\times 2^9$$

$$S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$$

$$= \sum_{j=1}^{10} j\,(j-1)\times\frac{10}{j}\times\frac{9}{j-1}\times {}^8C_{j-2}$$

$$\Rightarrow \quad S_2 = 10\times 9\sum_{j=1}^{10} {}^8C_{j-2} = 90\times 2^8$$

$$= \sum_{j=1}^{10} j^2\,{}^{10}C_j = \sum_{j=1}^{10} \{j(j-1)+j\}^{10}C_j$$

$$= \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j + \sum_{j=1}^{10} j\,{}^{10}C_j$$

$$= 10\times 9\times 2^8 + 10\times 2^9$$

$$= 11\times 10\times 2^8$$

So, Statement II is true and Statement I is false.

6. ***(a)*** $\sum_{j=0}^{n} a^{n-j}\cdot b^j = a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + ab^{n-1} + b^n$

$$= \frac{a^n[1-(b/a)^{n+1}]}{1-b/a} \qquad \text{[sum of a GP]}$$

$$= \frac{a^{n+1}-b^{n+1}}{a-b}$$

Also, coefficient of x^m in $(\alpha x+\beta)^{m+1}$ is $(m+1)\alpha^m\beta$.

$\therefore$ Statement II is true.

Next, $\sum_{j=0}^{n} (-1)^j(2x+3)^{n-j}(5-2x)^j$

$$= \sum_{j=0}^{n} (2x+3)^{n-j}(2x-5)^j$$

$$= \frac{(2x+3)^{n+1}-(2x-5)^{n+1}}{(2x+3)-(2x-5)}$$

$$= \frac{1}{8}[(2x+3)^{n+1}-(2x-5)^{n+1}]$$

Coefficient of x^n

$$= \frac{1}{8}[{}^{n+1}C_n\,2^n(3) - {}^{n+1}C_n\,2^n(-5)] = (n+1)2^n$$

$\therefore$ Statement I is true and Statement II is correct explanation for it.

7. ***(c)*** $C_0 + C_1\omega + C_2\omega^2 + C_3 + C_4\omega + C_5\omega^2 + \ldots$

$$= \sum_{k=0}^{3n} C_k\omega^k = (1+\omega)^{3n}$$

$$= (-\omega^2)^{3n} = (-1)^{3n}\omega^{6n}$$

$$= (-1)^n(1) = (-1)^n$$

$\therefore$ Statement II is false, as area of triangle formed by cube roots of unity is $\frac{3\sqrt{3}}{4}$ sq units.

8. ***(a)*** $C_0^2 + C_1^2 + \ldots + C_n^2$

$$= C_0C_n + C_1C_{(n-1)} + \ldots + C_nC_0$$

$$= {}^{2n}C_n$$

$\therefore$ Statement II is true.

Let $\quad S = 5C_0^2 + 7C_1^2 + 9C_2^2 + \ldots + (5+2n)C_n^2 \qquad \ldots(i)$

Using $C_r = C_{n-r}$, we can rewrite Eq. (i) as

$$S = (5+2n)\,C_0^2 + (3+2n)C_1^2 + \ldots + 5C_n^2 \qquad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2S = (10+2n)(C_0^2 + C_1^2 + \ldots + C_n^2)$$

$$\Rightarrow \quad S = (5+n)({}^{2n}C_n) = (5+n)\frac{2n}{n!n!}$$

9. ***(a)*** $C_0 = {}^{n-1}C_0$

$$-C_1 = -{}^{n-1}C_0 - {}^{n-1}C_1$$

$$C_2 = {}^{n-1}C_1 + {}^{n-1}C_2$$

$$-C_3 = -{}^{n-1}C_2 - {}^{n-1}C_3$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

$$(-1)^{m-1}C_{m-1} = (-1)^{m-1}({}^{n-1}C_{m-2} + {}^{n-1}C_{m-1})$$

On adding the above equations, we get

$$C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^{m-1}C_{m-1} = (-1)^{m-1}({}^{n-1}C_{m-1})$$

10. ***(b)*** Given, $(1+t)^n = C_0 + C_1t + C_2t^2 + \ldots + C_nt^n \qquad \ldots(i)$

From Eq. (i),

$$\int_0^x (1+t)^n\,dt = \int_0^x [C_0 + C_1t + C_2t^2 + \ldots + C_nt^n]\,dt$$

$$= \frac{1}{n+1}[(1+x)^{n+1} - 1]$$

$$= \frac{C_0}{1}x + \frac{C_2}{2}x^2 + \ldots + \frac{C_n}{n+1}x^{n+1} \qquad \ldots(ii)$$

On multiplying Eq. (i) by t and integrating, we get

$$\int_0^x t(1+t)^n\,dt = \int_0^x [C_0t + C_1t^2 + \ldots + C_nt^{n+1}]\,dt$$

$$= \frac{x(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+1)(n+2)}$$

$$= \frac{C_0}{2}x^2 + \frac{C_1}{3}x^3 + \frac{C_2}{4}x^4 + \ldots + \frac{C_n}{n+2}x^{n+2} \qquad \ldots(iii)$$

Putting $x = -1$ in Eq. (iii), we obtain that the second statement is true.

Putting $x = 1$ in Eqs. (ii) and (iii) and subtracting, we obtain that Statement I is also true.

11. ***(d)*** Statement II is false can be seen from theory.

We can write

$$(\sqrt{3}+\sqrt{2})^{50} = 2^{25}\left(1+\frac{\sqrt{3}}{2}\right)^{50}$$

Let $\quad k = \dfrac{(50+1)\sqrt{3/2}}{1+\sqrt{3/2}}$

$$= \frac{51\sqrt{3}(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$= 51(3-\sqrt{6}) = 51(0.5505)$$

$$= 28.07$$

$$\Rightarrow \quad [k] = 28$$

Thus, the greatest term in the expansion of $(\sqrt{3}+\sqrt{2})^{50}$ is the 29th term and it equals

$$2^{25}\binom{50}{28}\left(\sqrt{\frac{3}{2}}\right)^{28} = \binom{50}{22}3^{14}2^{11}$$

12. ***(d)*** We have,

$$\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{n}=\binom{n+1}{m} \quad \ldots(i)$$

LHS of Eq. (i)

= Coefficient of x^m in

$$\{(1+x)^m+(1+x)^{m+1}+\ldots+(1+x)^n\}$$

[using Statement II]

$$= \text{Coefficient of } x^m \text{ in } (1+x)^m \frac{\{(1+x)^{n-m+1}-1\}}{1+x-1}$$

$$= \text{Coefficient of } x^{m+1} \text{ in } [(1+x)^{n+1}-(1+x)^m]$$

$$= \binom{n+1}{m+1}$$

13. ***(d)*** Let $S=\sum_{r=0}^{n} C_r \sin(rx)[\cos(n-r)x] \quad \ldots(i)$

Using $C_r = C_{n-r}$, we can write

$$S=\sum_{r=0}^{n} C_{n-r}\sin(rx)\cos\{(n-r)x\}$$

$$=\sum_{r=0}^{n} C_r \sin[(n-r)x]\cos(rx) \quad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2S=\sum_{r=0}^{n} C_r[\sin(rx)\cos\{(n-r)x\}+\sin\{(n-r)x\}\cos(rx)]$$

$$=\sum_{r=0}^{n} C_r \sin(nx)=\sin(nx)\sum_{r=0}^{n} C_r$$

$$=2^n \sin(nx)$$

$$\Rightarrow \quad S=2^{n-1}\sin(nx)$$

14. ***(a)*** $^{10}C_0+(^{10}C_0+{}^{10}C_1)+(^{10}C_0+{}^{10}C_1+{}^{10}C_2)$

$$+\ldots+(^{10}C_0+{}^{10}C_1+{}^{10}C_2+\ldots+{}^{10}C_9)$$

$$=10\,{}^{10}C_0+9\,{}^{10}C_1+8\,{}^{10}C_2+\ldots+{}^{10}C_9$$

$$={}^{10}C_1+2\,{}^{10}C_2+3\,{}^{10}C_3+\ldots+10\,{}^{10}C_{10}$$

$$=\sum_{r=1}^{10} r\,{}^{10}C_r=10\sum_{r=1}^{10}{}^{9}C_{r-1}=10\cdot 2^9$$

15. ***(a)*** $3456^{2222}=(7\times 493+5)^{2222}$

$$=(7k+5)^{2222}$$

$$=7m+5^{2222}$$

Now, $5^{2222}=5^2(5^3)^{740}$

$$=25(125)^{740}=25(126-1)^{740}$$

$$=25(7n+1)=175n+25$$

Remainder when $175n+25$ is divided by 7, is 4.

Hence, both the Statements are correct and Statement II is a correct explanation of Statement I.

16. ***(a)*** $(1+x+x^2)^n=\sum_{r=0}^{2n} a_r x^r \quad \ldots(i)$

We know that,

$$(1-x)^n=\sum_{r=0}^{n}(-1)^{n-r}\,{}^nC_r x^r$$

$$=\sum_{r=0}^{n}(-1)^{n-r}\,{}^nC_r x^{n-r} \quad \ldots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$\sum_{r=0}^{n}(-1)^{n-r}\,{}^nC_r a_r = \text{Coefficient of } x^n \text{ in } (1-x^3)^n$$

Since, $n \neq 3k$, we have

$$\sum_{r=0}^{n}(-1)^{n-r}a_r\,{}^nC_r=0 \text{ or } \sum_{r=0}^{n}(-1)^r a_r\,{}^nC_r=0$$

Hence, both the Statements are correct and Statement II is a correct explanation of Statement I.

17. ***(b)*** $\frac{T_{r+1}}{T_r}=\frac{12-r+11}{r}\cdot\frac{11}{10}$

Let $T_{r+1}\geq T_r$ or $13-r\geq 1.1x$

or $13\geq 2.1r$ or $r\leq 6.19$

So, the greatest term occurs for $r=6$. Hence, 7th term is the greatest term. Also, the binomial coefficient of 7th term is $^{12}C_6$ which is the greatest binomial coefficient. But this is not the reason for which T_7 is the greatest.

Here, it is coincident that the greatest term has the greatest binomial coefficient.

Hence, Statement I is true, Statement II is true; but Statement II is not a correct explanation of Statement I.

18. ***(d)*** Statement II is true as it is the property of binomial coefficients. But Statement I is false as three consecutive binomial coefficient may be in AP but not always.

19. ***(b)*** $(1+x+x^2+x^3+x^4)^{1000}$

$$=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_{4000}x^{4000}$$

Clearly, there are 4001 terms. Also, the number of terms in the expansion $(a_1+a_2+\ldots+a_m)^n$ is $^{n+m-1}C_{m-1}$.

Clearly, Statement II has nothing to do with Statement I.

20. ***(b)*** We know that, the total number of terms in $(x_1+x_2+\ldots+x_r)^n$ is $^{n+r-1}C_{r-1}$.

So, the total number of terms in $(x_1+x_2+\ldots+x_n)^3$ is

$$^{3+n-1}C_{n-1}={}^{n+2}C_{n-1}={}^{n+2}C_3$$

$$=\frac{(n+2)(n+1)n}{6}$$

and total number of terms in $(x_1+x_2+x_3)^n$ is $^{n+3-1}C_{n-1}$

$$={}^{n+2}C_3$$

$$=\frac{(n+2)(n+1)n}{6}$$

21. ***(a)*** $(1+x)^n-nx-1=(1+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n)$

$$-nx-1 \quad \ldots(i)$$

$$={}^nC_2x^2+\ldots+{}^nC_nx^n$$

$$=x^2({}^nC_2+{}^nC_3x+\ldots+{}^nC_nx^{n-2})$$

Hence, $(1+x)^n-nx-1$ is divisible by x^2.

Now, in Eq. (i), replace x by 8 and n by $(n+1)$.

Then, we have $(1+8)^{n+1}-(n+1)8-1$

$$=8^2({}^nC_2+{}^nC_3 8+\ldots+{}^nC_n 8^{n-2})$$

or $(9)^{2n+2}-8n-9=8^2({}^nC_2+{}^nC_3 8+\ldots+{}^nC_n 8^{n-2})$

which is divisible by 64.

Hence, both the Statements are correct and Statement II is a correct explanation of Statement I.

22. ***(a)*** Coefficient of x^n in $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}\right)^3$

$= $ Coefficient of x^n in $\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\right)^3$

[as higher powers of x are not counted while calculating the coefficient of x^n]

$= $ Coefficient of x^n in $e^{3x} = \frac{3^n}{n!}$

23. ***(b)*** $(1 + x)^{41}(1 - x + x^2)^{40}$

$= (1 + x)(1 + x)^{40}(1 - x + x^2)^{40}$

$= (1 + x)(1 + x^3)^{40}$

$= (1 + x^3)^{40} + x(1 + x^3)^{40}$

$= (1 + {}^{40}C_1x^3 + {}^{40}C_2x^6 + \ldots + {}^{40}C_{40}x^{120})$

$+ ({}^{40}C_0 + {}^{40}C_1x^4 + {}^{40}C_2x^7 + \ldots + {}^{40}C_{40}x^{121})$

Hence, the coefficient of x^{85} is zero as there is no term in the above expansion which has x^{85}.

Also, Statement II is correct but it is not a correct explanation of Statement I.

24. ***(a)*** $S = \sum_{0 \le i} \sum_{< j \le n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right)$

$= \sum_{0 \le i} \sum_{< j \le n} \left(\frac{n - i}{{}^nC_{n-i}} + \frac{n - j}{{}^nC_{n-j}}\right)$

$= n \sum_{0 \le i} \sum_{\le j \le n} \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right) - S$

$= \frac{n}{2} \sum_{0 \le i} \sum_{\le j \le n} \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right)$

$= \frac{n}{2}\left(\sum_{r=0}^{n-1} \frac{n - r}{{}^nC_r} + \sum_{r=1}^{n} \frac{r}{{}^nC_r}\right)$

$= \frac{n}{2}\left(\sum_{r=0}^{n} \frac{n}{{}^nC_r}\right)$

$= \frac{n^2}{2}a$

25. ***(a)*** We have,

$(2 + \sqrt{5})^P + (2 - \sqrt{5})^P = 2[2^P + {}^PC_2\,2^{P-5}5$

$+ {}^PC_4\,2^{P-4}5^2 + \ldots + {}^PC_{P-1}\,2 \times 5^{(P-1)/2}]$ ...(i)

From Eq. (i), $(2 + \sqrt{5})^P + (2 - \sqrt{5})^P$ is an integer and

$-1 < (2 - \sqrt{5})^P < 0$ [$\because P$ is odd]

So, $[(2 + \sqrt{5})^P] = (2 + \sqrt{5})^P + (2 - \sqrt{5})^P$

$= 2^{P+1} + {}^PC_2\,2^{P-1}5 + \ldots + {}^PC_{P-1}\,2^2 5^{(P-1)/2}$

$\therefore \quad [(2 + \sqrt{5})^P] - 2^{P+1} = 2[{}^PC_2\,2^{P-2}5$

$+ {}^PC_4\,2^{P-4}5^2 + \ldots + {}^PC_{P-1}\,2 \times 5^{(P-1)/2}]$

Now, all the binomial coefficients

$= {}^PC_2 = \frac{P(P-1)}{1 \times 2}$

$${}^PC_4 = \frac{P(P-1)(P-2)(P-3)}{1 \times 2 \times 3 \times 4}, \ldots, \quad {}^PC_{P-1} = P$$

are divisible by the prime P. Thus, RHS is divisible by P.

26. ***(a)*** We know that,

${}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^nC_r$

$= $ Coefficient of x^r in $(1 + x)^m(1 + x)^n$

$= $ Coefficient of x^r in $(1 + x)^{m+n}$

$= {}^{m+n}C_r$

$= 0$, as $m + n < r$

27. ***(b)*** Obviously, Statement II is true. But to get the sum of coefficients in the expansion of $(3^{-x/4} + 3^{5x/4})^n$, we must put $x = 0$.

28. ***(a)*** Statement II is true (can be checked easily) and hence

${}^{2n}C_0 < {}^{2n}C_1 < {}^{2n}C_2 < \ldots < {}^{2n}C_{n-1}$

$< {}^{2n}C_n > {}^{2n}C_{n+1} > \ldots > {}^{2n}C_{2n}$.

29. ***(a)*** $\left(x + \frac{9}{x} + 6\right)^n = \left(\frac{x^2 + 6x + 9}{x}\right)^n = \frac{(3 + x)^{2n}}{x^n}$

$= \frac{3^{2n}}{x^n}\left(1 + \frac{x}{3}\right)^{2n}$

$\therefore$ Coefficient of the term independent of x, $\left(x + \frac{9}{x} + 6\right)^n$

$= 3^{2n}\left[\text{Coefficient of } x^n \text{ in the expansion of } \left(1 + \frac{x}{3}\right)^{2n}\right]$

$= 3^{2n}\binom{2n}{n}\left(\frac{1}{3}\right)^n$

$= \frac{3^n(2n)!}{n!n!}$ [using Statement II]

30. ***(a)*** For Statement II,

Write $n = 7r + k$; $k = -3, -2, -1, 0, 1, 2, 3$

Now, $n^7 - n = (7r + k)^7 - (7r + k)$

Using binomial theorem, we get

$n^7 - n - (k^7 - k)$ is divisible by 7.

$\Rightarrow$ $n^7 - n$ is divisible by 7 if and only if $k^7 - k$ is divisible by 7.

For $k = \pm 3, \pm 2, \pm 1$, $k^7 - k$ is divisible by 7 if and only if $k^6 - 1$ is divisible by 7.

But $k^6 - 1 = 728, 63, 0$ for $k = \pm 3, \pm 2, \pm 1$.

$\therefore$ $k^6 - 1$ is divisible by 7 for $k = \pm 3, \pm 2, \pm 1$.

Thus, $n^7 - n$ is divisible by 7 for each $n \in N$.

$\Rightarrow (n + 1)^7 - (n + 1) - (n^7 - n)$ is divisible by 7.

$\Rightarrow (n + 1)^7 - n^7 - 1$ is divisible by 7.

Therefore, both the Statements are true and Statement II is a correct explanation of Statement I.

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JEE ADVANCED DRILL

A collection of 30 innovative problems on JEE Advanced pattern from complete syllabus

Single Option Correct Type

1. If the roots of the cubic $x^3+ax^2+bx+c=0$ are three consecutive positive integers, then the value of $\frac{a^2}{b+1}$ is equal to

(a) 3 *(b)* 2 *(c)* 1 *(d)* 1/3

2. If $f(x)=\sin x$, $g(x)=[x+1]$ and $g(f(x))=h(x)$, where $[\cdot]$ is the greatest integer function, then $h'\left(\frac{\pi}{2}\right)$ is

(a) does not exist *(b)* 1
(c) -1 *(d)* None of these

3. If $x\cos\alpha+y\sin\alpha=p$ a variable chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{2a^2}=1$ subtends a right angle at the centre of hyperbola, then the chord touch a fixed circle, whose radius is equal to

(a) $\sqrt{2}a$ *(b)* $\sqrt{3}a$
(c) $2a$ *(d)* $\sqrt{5}a$

4. Suppose that $f(x)$ is a differentiable function such that $f'(x)=g'(x)$ exists and $|f(x)|\le 1$ for all real x, if $\{f(0)\}^2+\{g(0)\}^2=9$ for some $c\in(-3,3)$, then

(a) $g(c)\,g''(c)>0$ *(b)* $g(c)\,g''(c)<0$
(c) $g(c)\,g''(c)\ge 0$ *(d)* $g(c)\,g''(c)\le 0$

5. The sum of the series

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}\right)+\ldots\infty \text{ is}$$

(a) $\frac{\pi}{6}$ *(b)* $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

One or More than One Option Correct Type

6. $\frac{1}{1!(n-1)!}-\frac{1}{3!(n-3)!}+\frac{1}{5(n-5)!}-\frac{1}{7(n-7)!}+\ldots$ is equal to

(a) $\frac{-2^n}{n!}$ for some values of n only
(b) $\pm\frac{2^{n-1}}{n!}$ for odd values of n only
(c) $\frac{2^n}{n!}$ for some even values of n only
(d) None of the above

7. The equation $\sin x=[1+\sin x]+[1-\cos x]$ has, where $[\cdot]$ is the greatest integer function

(a) no solution in $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ *(b)* no solution in $\left[\frac{\pi}{2},\pi\right]$
(c) no solution in $\left[\pi,\frac{3\pi}{2}\right]$ *(d)* no solution for $x\in R$

8. If $f(x)$, $g(x)$ be twice differential function on [0, 2] satisfying $f''(x)=g''(x)$, $f'(1)=2g'(1)=4$ and $f(2)=3g(2)=9$, then

(a) $f(4)-g(4)=10$
(b) $|f(x)-g(x)|<2\Rightarrow -2<x<0$
(c) $f(2)=g(2)\Rightarrow x=-1$
(d) $f(x)-g(x)=2$ has real roots

9. Line $\frac{x}{a}+\frac{y}{b}=1$ cuts the coordinates axes at $A(a,0)$ and $B(0,b)$ and the line $\frac{x}{a'}+\frac{y}{b'}=-1$ at $A(-a',0)$ and $B(0,-b')$. If the points $AB\,A'\,B'$ are concylic, then the orthocentre of $\Delta ABA'$ is

(a) (0, 0) *(b)* $(0, b')$
(c) $\left(0,\frac{aa'}{b}\right)$ *(d)* $\left(0,\frac{bb'}{a}\right)$

10. z is complex number such that $|z-2+2i|=1$, then
(a) its least modulus is $2\sqrt{2}-1$
(b) its least argument is $\pi/4$
(c) its arguments, when modulus is least is $-\pi/4$
(d) its greatest argument is $\pi/4$

11. In an experimental performance of a single throw of a pair of unbiased normal dice, let three events E_1, E_2 and E_3 are defined as
E_1 = getting prime numbered face on each dice
E_2 = getting the same number on each dice
E_3 = getting total on two dice equal to 4
Which of the following is/are true?
(a) The probabilities $P(E_1)$, $P(E_2)$, $P(E_3)$ are in AP
(b) The events E_1 and E_2 are independent
(c) $P(E_3/E_1)=2/9$
(d) $P(E_1+E_2)+P(E_2-E_3)=17/36$

12. If a, b and c are distinct positive real number such that bc, ca, ab are in GP, then
(a) $b^2>\dfrac{2a^2c^2}{a^2+c^2}$ *(b)* $b^2<\dfrac{2a^2c^2}{a^2+c^2}$
(c) $b^2<\dfrac{a^2+c^2}{2}$ *(d)* $b^2>\dfrac{a^2+c^2}{2}$

13. Let $f(x)=x^3\,f(1)+x^2f'(2)+xf''(3)+f'''(4)$, then
(a) $f(3)-f(1)=0$ *(b)* $f'(2)=0$
(c) $f''(3)=0$ *(d)* $f'''(4)=0$

Single Integer Type

14. Let the normals are drawn from (α, β) to the hyperbola $xy=1$ and (x_i, y_i), $i=1, 2, 3, 4$, be the fact of the co-normal points. If the algebraic sum of the perpendicular distances drawn from (x_i, y_i), $i=1, 2, 3, 4$ onto a variable line vanishes, then the variable line passes through the point $\left(\dfrac{\alpha}{\gamma}, \dfrac{\beta}{\gamma}\right)$, then the value of γ is ..... .

15. In ΔABC, if $\tan\dfrac{A}{2}$, $\tan\dfrac{B}{2}$, $\tan\dfrac{C}{2}$ are in HP, then the minimum value of $\cot^2 B/2$ is ..... .

16. Let f be a real valued function satisfying $f(xy)=f(x)+f(y)$, $f'(1)=-3$. Then, the area bounded by $y=f(x)$ and the coordinates axes is ..... .

17. A is 4×4 matrix with $a_{11}=1+x_1$, $a_{22}=1+x_2$, $a_{33}=1+x_3$, $a_4=1+x_4$ and all other entries 1, where x_i are the roots of $x^4-x^2+1=0$. The value of det (A) is ..... .

18. If $f(x)$ is a differentiable and strictly increasing function $f''(x)>0$, then the value of $\lim\limits_{x\to0}\dfrac{f(x)-f(x^2)}{f(x)-f(0)}$ is ..... .

19. Thirty-two players ranked 1 to 32 are playing in a knaekant tournament. Assume that in every match between any two players the better ranked player wins, the probability that ranked 1 and ranked 2 players are winner and runner up respectively is p, then the value of $[2/p]$ is (where, $[\cdot]$ represents the greatest integer function) ..... .

20. If S denotes the sum of the series $\dfrac{3}{2^3}+\dfrac{4}{2^4\cdot3}+\dfrac{5}{2^6\cdot3}+\dfrac{6}{2^7\cdot5}+\ldots\infty$, then the value of S^{-1} is ..... .

21. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors such that $\mathbf{a}-\mathbf{b}=0=\mathbf{a}\cdot\mathbf{c}$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\dfrac{\pi}{3}$, then find the value of $|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|$.

Linked Comprehension Type

Directions (Q. Nos. 22-23) The asymptotes of a rectangular hyperbola are parallel to the coordinate axes. If two perpendicular tangents of the hyperbola intersect at (2, 2) and orthocentre of triangle formed by any three points on the hyperbola is (0, 0).

22. The equation of conjugate hyperbola is
(a) $(x-2)(y-2)=-4$ *(b)* $(x-3)(y-3)=9$
(c) $(x-2)(y-2)=4$ *(d)* $(x-4)(y+4)=-16$

23. A tangent to the hyperbola at (4, 4) intersects the asymptote of points A and B, then the length AB is
(a) 4 *(b)* $4\sqrt{2}$ *(c)* $8\sqrt{2}$ *(d)* 16

Directions (Q. Nos. 24-25)
Let $a_k=k\,({}^8C_k)$, $b_k=(8-k)\,{}^8C_k$ and $A_k=\begin{bmatrix}a_k & 0\\ 0 & b_k\end{bmatrix}$. Let $A=\sum\limits_{k=1}^{7}A_k=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$.

24. The value of $a+b$ is
(a) 2032 *(b)* 2040 *(c)* 1016 *(d)* 2010

25. $\dfrac{ab}{2^7-1}$ is equal to
(a) 1016 *(b)* 8128
(c) 2032 *(d)* None of these

Directions (Q. Nos. 26-27) $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of a ΔABC inscribed in the circle $|z|=2$. Internal angle bisector of the angle A meets the circumcircle again at $D(z_4)$.

26. Complex number representing point D is
(a) $z_4=\dfrac{1}{z_2}+\dfrac{1}{z_3}$ *(b)* $\sqrt{\dfrac{z_2+z_3}{z_1}}$ *(c)* $\sqrt{\dfrac{z_2z_3}{z_1}}$ *(d)* $z_4=\sqrt{z_2z_3}$

27. $\arg[z_4/(z_2-z_3)]$ is equal to
(a) $\dfrac{\pi}{4}$ *(b)* $\dfrac{\pi}{3}$ *(c)* $\dfrac{\pi}{2}$ *(d)* $\dfrac{2\pi}{3}$

Match the Columns

28. Match the following columns.

	Column I	Column II
A.	If the vertices of a rectangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are extremities of latusrectum, then the eccentricity of the ellipse is	p. $\frac{2}{\sqrt{5}}$
B.	If the extremities of the diameter of circle $x^2+y^2=16$ are the foci of the ellipse, then the eccentricity of the ellipse, when its size is just sufficient to contain the circle, is	q. $\frac{1}{\sqrt{2}}$
C.	If the normal at point (6, 2) to the ellipse passes through its nearest focus (5, 2), having centre at (4, 2), then its eccentricity is	r. $\frac{1}{3}$
D.	If the extremities of the latusrectum of the parabola $y^2=24x$ are the foci of ellipse and if the ellipse passes through the vertex of the parabola, then its eccentricity is	s. $\frac{1}{2}$

29. Match the following columns.

	Column I	Column II
A.	If $\int \frac{2^x}{\sqrt{1-4^x}}dx = k\sin^{-1}[f(x)]+C$, then k is greater than	p. 0
B.	If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}dx = a\ln\frac{x^k}{x^{k+1}}+C$, then ak is less than	q. 1
C.	If $\int \frac{x^4+1}{x(x^2+1)^2}dx = k\ln(x)+\frac{m}{1+x^2}+n$, where n is the constant at integration, then mk is greater than	r. 3
D.	If $\int \frac{dx}{5+4\cos x} = k\tan^{-1}\left(m\tan\frac{x}{2}\right)+C$, then k/m is greater than	s. 4

30. In acute angled ΔABC,

	Column I	Column II
A.	$\cos A, \cos B, \cos C$ are in AP.	p. Distance of orthocentre from vertices of triangle are in AP.
B.	$\sin(A/2), \sin(B/2), \sin(C/2)$ are in AP.	q. Distance of orthocentre from sides of triangle are in AP.
C.	Distances of circumcentre from the vertices of the ΔABC are in AP.	r. Distance of incentre from vertices of triangle are in HP.
D.	Circumradii of ΔOBC, ΔOAC and ΔOAB are in HP (where, O is the circumcentre of ΔABC).	s. Distance of incentre from excentres of triangle are in AP.

Answers with Explanation

1. ***(a)*** We have, $x^3+ax^2+bx+c=0$ have roots are three consecutive positive integers.

Let three consecutive positive integer be $r, r+1, r+2$.

Where, a = Sum of roots $= r+r+1+r+2=3r+3$

and b = Product of roots taken of two

$= r(r+1)+(r+1)(r+2)+r(r+2)$

$= 3r^2+6r+2$

Now, $\frac{a^2}{b+1}=\frac{(3r+3)^2}{3r^2+6r+2+1}=\frac{9(r+1)^2}{3(r^2+2r+1)}=3$

2. ***(a)*** We have,

$f(x)=\sin x,\ g(x)=[x+1]$

$h(x)=g(f(x))=[f(x)+1]=[\sin x+1]$

$\therefore\quad h\left(\frac{\pi}{2}\right)=\left[\sin\frac{\pi}{2}+1\right]=[1+1]=2$

Now, $\text{LHD}=\lim_{k\to 0^+}\frac{h\left(\frac{\pi}{2}-k\right)-h\left(\frac{\pi}{2}\right)}{-k}$

$=\lim_{k\to 0^+}\frac{\left[\sin\left(\frac{\pi}{2}-k\right)+1\right]-2}{-k}$

$=\lim_{k\to 0^+}\frac{[\cos k+1]-2}{-k}$

$=\lim_{k\to 0^+}\frac{[\cos k-1]}{-k}=\infty$

$\therefore$ LHD does not exist.

3. ***(a)*** Homogenising the equation of hyperbola with the help of line.

We have, $\frac{x^2}{a^2}-\frac{y^2}{2a^2}=\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2$

$\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{2a^2}=\frac{x^2\cos^2\alpha}{p^2}+\frac{y^2\sin^2\alpha}{p^2}+\frac{2xy\sin\alpha\cos\alpha}{p^2}$

$\Rightarrow x^2\left(\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}\right)-y^2\left(\frac{1}{2a^2}+\frac{\sin^2\alpha}{p^2}\right)-\frac{2xy\sin\alpha\cos\alpha}{p^2}=0$

Now, this subtends an angle of 90° at origin.

$\therefore$ Coefficient of x^2 + Coefficient of $y^2=0$

$\Rightarrow \frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}-\frac{1}{2a^2}-\frac{\sin^2\alpha}{p^2}=0$

$\Rightarrow \frac{1}{2a^2}-\frac{1}{p^2}=0\ \Rightarrow\ p=\sqrt{2}a$

Here, p is radius of circle, whose touches the chord

$x\cos\alpha+y\sin\alpha=p$.

Hence, the radius of circle is $\sqrt{2}a$.

4. ***(b)*** Let $h(x)=\{f(x)\}^2+\{g(x)\}^2$

By LMVT, there exists $c_1\in(-3,0)$, $c_2\in(0,3)$ such that $|g(c_1)|$ and $|g(c_2)|$ both are $\le 2/3$.

Hence, $h(c_1), h(c_2)<9=h(0)$

This gives that $h(x)$ has a maximum at some point $c\in(-3,3)$.

But, $g(c)\{f(c)+g'(c)\}=0$. If $g(c)=0$, then $h(c)\le 1$, a contradiction, since $h(c)\ge 9$

Hence, $g(c)\neq 0$ and so $f(c)+g'(c)=0$

Since, $h(x)$ has a maximum at c, $h''(c)\le 0$

This implies that $g(c)\{g(c)+g''(c)\}\le 0$, $\{g(c)\}^2>0$

We get, $g(c)\,g''(c)<0$

5. ***(d)*** We have,

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+\ldots$$

$$T_n=\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$$

$$=\sin^{-1}\left[\frac{1}{\sqrt{n}}\sqrt{1-\left(\frac{1}{\sqrt{n+1}}\right)^2}-\frac{1}{\sqrt{n+1}}\sqrt{1-\left(\frac{1}{\sqrt{n}}\right)^2}\right]$$

$$=\sin^{-1}\left(\frac{1}{\sqrt{n}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$$

$$\therefore\quad S=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)+\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$-\sin^{-1}\left(\frac{1}{\sqrt{4}}\right)+\sin^{-1}\left(\frac{1}{\sqrt{4}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\ldots\infty$$

$$=2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$=2\times\frac{\pi}{4}=\frac{\pi}{2}$$

6. ***(a, b, c)*** Let $S=\dfrac{1}{1!(n-1)!}-\dfrac{1}{3!(n-3)!}+\dfrac{1}{5!(n-5)!}-\ldots$

$$\Rightarrow\quad S=\frac{1}{n!}\left[\frac{n!}{1!(n-1)!}-\frac{n!}{3!(n-3)!}+\frac{n!}{5!(n-5)!}-\ldots\right]$$

$$\Rightarrow\quad S=\frac{1}{n!}[{}^nC_1-{}^nC_3+{}^nC_5-{}^nC_7+\ldots]$$

where, $S=\dfrac{1}{n!}\text{Im}[(1+i)^n]$, such that

$$S=\frac{1}{n!}\text{Im}\left[\left\{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^n\right\}\right]$$

$$\Rightarrow\quad S=\frac{1}{n!}\text{Im}\left[2^{n/2}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)\right]$$

$$\therefore\quad S=\frac{2^{n/2}}{n!}\sin\frac{n\pi}{4}$$

$$=\begin{cases}\pm\dfrac{2^n}{n!}, & n=2k, k\in\text{odd}\\ 0 & n=2k,\ k\in\text{even}\\ \pm\dfrac{2^{n-1}}{n!}, & n=2k-1\end{cases}$$

7. ***(a, b, c, d)*** We have,

$$\sin x=[1+\sin x]+[1-\cos x]$$

$$\Rightarrow\quad \sin x=2+[\sin x]+[-\cos x]$$

At $x=-\dfrac{\pi}{2}$, $\sin\left(-\dfrac{\pi}{2}\right)=2-1+0=-1\neq 1$

At $x=\dfrac{\pi}{2}$, $\sin\left(\dfrac{\pi}{2}\right)=2+\left[\sin\dfrac{\pi}{2}\right]+\left[-\cos\dfrac{\pi}{2}\right]$

$$=1\neq 3$$

At $x=\pi$, $\sin\pi=2+[\sin\pi]+[-\cos\pi]$

$$=0\neq 2+1$$

For $x\in\left(-\dfrac{\pi}{2},0\right)$,

$$\sin x=2+(-1)+0=1\qquad\text{[impossible]}$$

For $x\in\left(0,\dfrac{\pi}{2}\right)$,

$$\sin x=2+0+(-1)=1\qquad\text{[impossible]}$$

For $x\in\left(\dfrac{\pi}{2},\pi\right)$,

$$\sin x=2+0+0=2\qquad\text{[impossible]}$$

For $x\in\left(\pi,\dfrac{3\pi}{2}\right)$,

$$\sin x=2-1+0=1\qquad\text{[impossible]}$$

8. ***(a, b, c)*** We have,

$$f''(x)=g''(x)$$

On integrating both sides we get

$$f'(x)=g'(x)+C$$

Putting $x=1$, we get

$$f'(1)=g'(1)+C$$

$$4=2+C\qquad[\because f'(1)=2g'(1)=4]$$

$$\Rightarrow\quad C=2$$

$$\therefore\quad f'(x)=g'(x)+2$$

Again, integrating w.r.t. x, we get

$$f(x)=g(x)+2x+C_1$$

Putting $x=2$, we get

$$9=3+4+C_1\qquad[\because f(2)=3g(2)=9]$$

$$C_1=2$$

$$\therefore\quad f(x)=g(x)+2x+2$$

Putting $x=4$, we get

$$f(4)-g(4)=10$$

$$|f(x)-g(x)|=|2x+2|<2$$

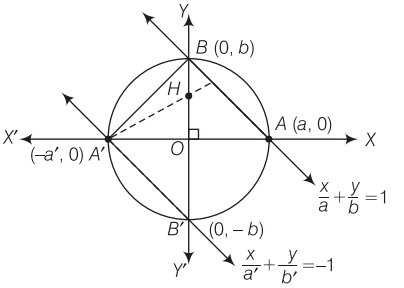
$$\Rightarrow\quad x\in(-2,0)$$

Also, $f(2)=g(2)\Rightarrow x=-1$

Hence, $f(x)-g(x)=2x$ has no solution.

9. ***(b, c)*** Equation of line OB is

$$x=0\qquad\ldots(i)$$



and equation of line AB is

$$\frac{x}{a}+\frac{y}{b}=1.$$

$\because A'M$ is perpendicular to the line AB.

Hence, equation of $A'M$ is

$$\frac{x}{b}-\frac{y}{a}=\lambda.$$

$\because A'M$ is passes through $(-a', 0)$.

$$\therefore \quad -\frac{a'}{b} - 0 = \lambda \Rightarrow \lambda = -\frac{a'}{b}$$

Equation of $A'B$ is

$$\frac{x}{b} - \frac{y}{a} = -\frac{a'}{b} \qquad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = 0, y = \frac{aa'}{b}$$

$\therefore$ Coordinates of H is $\left(0, \frac{aa'}{b}\right)$.

Since, $ABA'B'$ are concyclic.

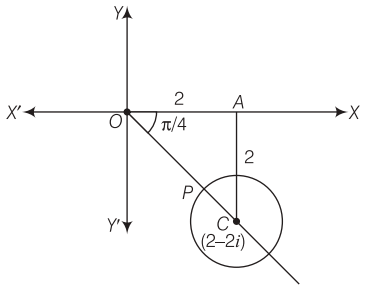
$$\therefore \quad OA \cdot OA' = OB \cdot OB'$$

$$\Rightarrow \quad -aa' = -bb' \Rightarrow \frac{aa'}{b} = b'$$

$\therefore$ Coordinates of H is $(0, b')$.

10. ***(a, c)*** Let the line joining the origin and the centre of the circle at P and Q as shown in the figure. Then, P is the point on the circle corresponding to the least modulus.

Its modulus $= OC - PC = 2\sqrt{2} - 1$



Then, the argument corresponding to the complex number represented by M is

$$\angle AOC = -\frac{\pi}{4} \qquad \left[\because \tan \angle AOC = \frac{2}{2} = 1\right]$$

11. ***(a, d)*** We have,

$$P(E_1) = \frac{1}{4}, P(E_2) = \frac{1}{6}, P(E_3) = \frac{1}{12}$$

(a) Now, $P(E_1) + P(E_3) = \frac{1}{4} + \frac{1}{12} = \frac{2}{6} = 2P(E_2)$

$\therefore$ They are in AP.

(b) $P(E_1 \cap E_2) = \frac{1}{12}, P(E_1 \cap E_3) = \frac{1}{36}, P(E_2 \cap E_3) = \frac{1}{36}$

Here, $P(E_1) \times P(E_2) \neq P(E_1 \cap E_2)$

$\therefore E_1$ and E_2 are not independent.

(c) $P\left(\frac{E_3}{E_1}\right) = \frac{P(E_3 \cap E_1)}{P(E_1)} = \frac{1/36}{1/12} = \frac{1}{3}$

(d) $P(E_1 + E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \frac{1}{4} + \frac{1}{6} - \frac{1}{12} = \frac{1}{3}$

$P(E_2 - E_3) = P(E_2 \cap E_3')$

$= P(E_2) - P(E_2 \cap E_3)$

$= \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$

$\therefore P(E_1 + E_2) + P(E_2 - E_3) = \frac{1}{3} + \frac{5}{36} = \frac{17}{36}$

12. ***(a, c)*** We have, bc, ca and ab are in GP.

$$\therefore \quad c^2a^2 = ab^2c \Rightarrow ac = b^2$$

and b^2c^2, c^2a^2 and a^2b^2 are also in GP.

We know that, AM > GM > HM

$$\therefore \quad \frac{b^2c^2 + a^2b^2}{2} > c^2a^2 > \frac{2}{\frac{1}{b^2c^2} + \frac{1}{a^2b^2}}$$

$$\Rightarrow \quad \frac{b^2c^2 + a^2b^2}{2} > b^4 > \frac{2b^2a^2c^2}{a^2 + c^2} \qquad [\because ac = b^2]$$

$$\Rightarrow \quad \frac{a^2 + c^2}{2} > b^2 > \frac{2a^2c^2}{a^2 + c^2}$$

13. ***(a, b, c, d)*** We have,

$$f(x) = x^3 f(1) + x^2 f'(2) + xf''(3) + f'''(4) \qquad \ldots(i)$$

$$f'(x) = 3x^2 f(1) + 2xf'(2) + f''(3) \qquad \ldots(ii)$$

$$f''(x) = 6xf(1) + 2f'(2) \qquad \ldots(iii)$$

$$f'''(x) = 6f(1) \qquad \ldots(iv)$$

Putting $x = 1, 2, 3, 4$ in Eqs. (i), (ii), (iii) and (iv), respectively, we get

$$f'(2) + f''(3) + f'''(4) = 0$$

$$12f(1) + 3f'(2) + f''(3) = 0$$

$$18f(1) + 2f'(2) - f''(3) = 0$$

$$6f(1) - f'''(4) = 0$$

$$\therefore \quad f(1) = f'(2) = f''(3) = f'''(4) = 0$$

So, $f(x)$ is a constant function.

14. ***(4)*** Equation of normal to the hyperbola $xy = 1$ at t is

$$xt^3 - yt - t^4 + 1 = 0 \qquad \ldots(i)$$

$\because$ It passes through (α, β), so $t^4 - \alpha t^3 + \beta t - 1 = 0$

If foot of the co-normal points are

$$\left(t_1, \frac{1}{t_1}\right), \left(t_2, \frac{1}{t_2}\right), \left(t_3, \frac{1}{t_3}\right) \text{ and } \left(t_4, \frac{1}{t_4}\right), \text{ then}$$

$$t_1 + t_2 + t_3 + t_4 = \alpha$$

$$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = \alpha$$

and $$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \beta$$

$$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = \beta$$

Let the variable line be

$$px + qy + r = 0 \qquad \ldots(ii)$$

Since, the algebraic sum of perpendicular distances from $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) is zero.

$$\Rightarrow p\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) + q\left(\frac{y_1 + y_2 + y_3 + y_4}{4}\right) + r = 0$$

$$\Rightarrow \quad \frac{p\alpha}{4} + \frac{q\beta}{4} + r = 0$$

Since, Eq. (ii) passes through $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right)$, so $\gamma = 4$

15. ***(3)*** We have, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in HP.

$$\therefore \quad \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2} \text{ are in AP.}$$

$$\Rightarrow \quad 2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2} \qquad \ldots(i)$$

and $$\cot\frac{B}{2}=\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{\cot\frac{A}{2}\cot\frac{C}{2}-1} \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\cot\frac{A}{2}\cot\frac{C}{2}=3$$

$$\Rightarrow \quad \cot\frac{B}{2}=\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{2}\geq\left(\cot\frac{A}{2}\cot\frac{C}{2}\right)^{1/2}\geq\sqrt{3}$$

$$\Rightarrow \quad \cot\frac{B}{2}\geq\sqrt{3} \quad \Rightarrow \quad \cot^2\frac{B}{2}\geq 3$$

Hence, the minimum value of $\cot^2\frac{B}{2}$ is 3.

16. ***(6)*** We have,

$$f(xy)=f(x)+f(y) \quad \ldots(i)$$

Hence, x and y are independent variable.

$$\therefore \quad f'(xy)\cdot y=f'(x)$$

Putting $x=1$ and $y=x$, we get

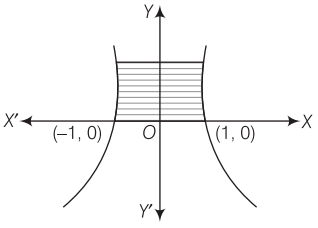
$$f'(x)x=f'(1)$$

$$\Rightarrow \quad f'(x)=-\frac{3}{x} \quad [\because f'(1)=-3]$$

On integrating both sides, we get

$$f(x)=-3\log|x|+C \quad \ldots(ii)$$

When $x=1$, then $f(1)=0$, from Eq. (ii), $C=0$



On putting the value of C in Eq. (ii), we get

$$f(x)=-3\log|x|$$

$$\therefore \text{Required area}=-3\int_{-1}^{1}\log|x|\,dx$$

$$=-6\int_{0}^{1}\log x\,dx$$

$$=-6[x\log x-1]_0^1$$

$$=-6(-1)=6 \text{ sq units}$$

17. ***(1)*** $A=\begin{vmatrix}1+x_1 & 1 & 1 & 1\\ 1 & 1+x_2 & 1 & 1\\ 1 & 1 & 1+x_3 & 1\\ 1 & 1 & 1 & 1+x_4\end{vmatrix}$

$$\therefore |A|=\Sigma x_1x_2x_3+x_1x_2x_3x_4=0+1=1$$

18. ***(1)*** $\because f(x)$ is strictly increasing function and $f''(x)>0$.

$\therefore f'(x)>0, \forall x\in$ domain of $f(x)$.

Now, $$\lim_{x\to 0}\frac{f(x)-f(x^2)}{f(x)-f(0)}$$

$$=\lim_{x\to 0}\frac{f'(x)-2xf'(x^2)}{f'(x)} \text{ [apply and L' Hospital's rule]}$$

$$=1-\lim_{x\to 0}\frac{2xf'(x^2)}{f'(x)}=1-2\times 0\times\frac{f'(0)}{f'(0)}=1-0=1$$

19. ***(3)*** For ranked 1 and 2 players to be winners and runners up, respectively, they should not be paired with each other in any round.

$$\therefore \quad p=\frac{30}{31}\times\frac{14}{15}\times\frac{6}{7}\times\frac{2}{3}=\frac{16}{31}$$

20. ***(2)*** Let $S=\sum_{r=1}^{\infty}\frac{r+2}{2^{r+1}\cdot r\cdot(r+1)}=\sum_{r=1}^{\infty}\frac{2(r+1)-r}{2^{r+1}\cdot r\cdot(r+1)}$

$$=\sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\left(\frac{2}{r}-\frac{1}{r+1}\right)=\sum_{r=1}^{\infty}\left(\frac{1}{2^r\cdot r}-\frac{1}{2^{r+1}(r+1)}\right)$$

$$=\lim_{n\to\infty}\left[\left(\frac{1}{2^1\times 1}-\frac{1}{2^2\times 2}\right)+\left(\frac{2}{2^2\times 2}-\frac{1}{2^3\times 3}\right)\right]$$

$$+\left(\frac{1}{2^3\times 3}-\frac{1}{2^4\times 4}\right)+\ldots+\left(\frac{1}{2^n\cdot n}-\frac{1}{2^{n+1}(n+1)}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2^{n+1}(n+1)}\right)=\frac{1}{2}$$

Hence, $S^{-1}=2$

21. ***(1)*** $\mathbf{a}\cdot\mathbf{b}=0=\mathbf{a}\perp\mathbf{b}$

$$\mathbf{a}\cdot\mathbf{c}=0\Rightarrow\mathbf{a}\perp\mathbf{c} \quad \Rightarrow \quad \mathbf{a}\perp\mathbf{b}-\mathbf{c}$$

$$|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|=|\mathbf{a}\times(\mathbf{b}-\mathbf{c})|$$

$$=|\mathbf{a}||\mathbf{b}-\mathbf{c}|=|\mathbf{b}-\mathbf{c}|$$

Now, $$|\mathbf{b}-\mathbf{c}|^2=|\mathbf{b}|^2+|\mathbf{c}|^2-2|\mathbf{b}||\mathbf{c}|\cos\frac{\pi}{3}$$

$$=2-2\times\frac{1}{2}=1$$

$$\Rightarrow \quad |\mathbf{b}-\mathbf{c}|=1$$

22. ***(a)*** Let the asymptotes of rectangular hyperbola be $x=a$ and $y=b$.

Given, two perpendicular tangents intersects at (2, 2).

$$\therefore \quad x=2, y=2$$

Equation of hyperbola is $(x-2)(y-2)+\lambda=0$.

Since, this hyperbola is passes through origin.

$$\therefore \quad (0-2)(0-2)+\lambda=0 \Rightarrow \lambda=-4$$

Hence, equation of hyperbola is $(x-2)(y-2)-4=0$.

Conjugate hyperbola is $(x-2)(y-2)+4=0$

$$\Rightarrow \quad (x-2)(y-2)=-4$$

23. ***(b)*** Tangent at (4, 4) is $y-4=-1(x-4)$

$$\Rightarrow \quad x+y=8$$

If $y=2$, then $x=6\Rightarrow A(6,2)$

If $x=2$, then $y=6\Rightarrow B(2,6)$

$$\therefore \quad AB=\sqrt{(6-2)^2+(2-6)^2}$$

$$=\sqrt{16+16}=\sqrt{32}=4\sqrt{2}$$

24. ***(a)*** We have, $a_k-k({}^8C_k)$, $b_k=(8-k)^8C_k$

$$A=\sum_{k=1}^{7}\begin{bmatrix}a_k & 0\\ 0 & b_k\end{bmatrix}=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$$

$$\therefore \quad a=\sum_{k=1}^{7}a_k \text{ and } b=\sum_{k=1}^{7}b_k$$

$$a+b=\sum_{k=1}^{7}(a_k+b_k)$$

$$=\sum_{k=1}^{7}8\ {}^8C_k=8(2^8-2)=8\times 254=2032$$

25. ***(b)*** We have, $a = \sum_{k=1}^{7} a_k = \sum_{k=1}^{7} k^8 C_k = \sum_{k=1}^{8} [k^8 C_k - 8^8 C_8]$

$= 8(2^7) - 8 \qquad \left[\because \sum_{r=1}^{n} r^n C_r = n\, 2^{n-1}\right]$

$= 8(2^7 - 1)$

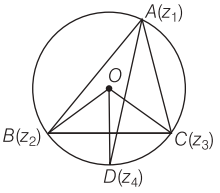
Similarly, $b = 8(2^7 - 1)$

$\therefore \quad ab = 64(2^7 - 1)^2$

Now, $\dfrac{ab}{2^7 - 1} = 64(2^7 - 1) = 64 \times 127 = 8128$

26. ***(d)*** Here, $\angle BOD = 2\angle BAD = A$

and $\angle COD = 2\angle CAD = A$



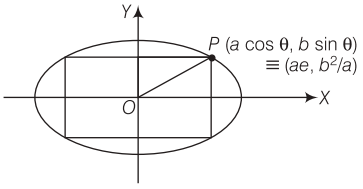
$\therefore \quad \dfrac{z_4}{z_2} = e^{iA}, \dfrac{z_3}{z_4} = e^{iA}$ [from rotation about the point O]

$\Rightarrow \quad z_4^2 = z_2 z_3$

27. ***(c)*** Clearly, OD bisects $\angle BAC$ of isosceles ΔBOC. Thus, angle between segments OD and BC is $\pi/2$.

$\therefore \quad \arg\left(\dfrac{z_4}{z_2 - z_3}\right) = \dfrac{\pi}{2}$

28. A $\rightarrow$ q; B $\rightarrow$ q; C $\rightarrow$ s; D $\rightarrow$ p



A. Let one of the vertices of the rectangle be $P(a\cos\theta, b\sin\theta)$. Then, its area is

$A = (2a\cos\theta)(2b\sin\theta) = 2ab\sin 2\theta$

Hence, $A_{max} = 2ab$

Now, the area of rectangle formed by the extremities of the latusrectum is

$$2ae \times \frac{2b^2}{a} = 4eb^2$$

Given that, $2ab = 4eb^2$

$\Rightarrow \quad \dfrac{2b}{a}e = 1$

$\Rightarrow \quad \dfrac{4b^2}{a^2}e^2 = 1$

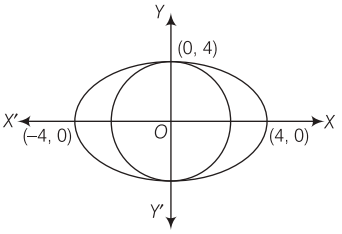
$\Rightarrow \quad 4(1 - e^2)e^2 = 1$

$\Rightarrow \quad 4e^4 - 4e^2 + 1 = 0$

$\Rightarrow \quad (2e^2 - 1)^2 = 0$

$\Rightarrow \quad e = \dfrac{1}{\sqrt{2}}$

B. For the ellipse, the distance c between the foci, $2ae = 8$ and the length of the semi-minor axis is $b = 4$.



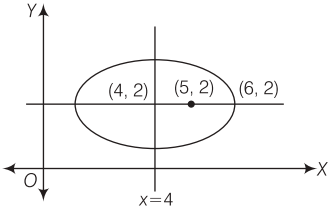
Now, $b^2 = a^2 - a^2e^2$

$\Rightarrow \quad 16 = a^2 - 16$

$\Rightarrow \quad a^2 = 32$

$\Rightarrow \quad e = \sqrt{1 - \dfrac{16}{32}} = \dfrac{1}{\sqrt{2}}$

C. Normal at point $P(6, 2)$ to the ellipse passes through its focus $Q(5, 2)$. Then, P must be the extremity of the major axis. Now, $ae = QR = 1$ (where, R is centre) and $a - ae = 1$.



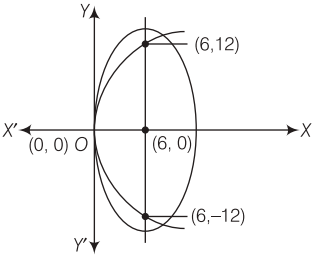
$\therefore \quad a = 2$

$b^2 = a^2 - a^2e^2$

$= 4 - 1 = 3$

$\Rightarrow \quad e = \sqrt{1 - \dfrac{3}{4}} = \dfrac{1}{2}$

D. Since, extremities of the latusrectum of the parabola $y^2 = 24x$ are $(6, \pm 12)$.



For ellipse, $2be = 24$ and the extremity of the minor axis is $(0, 0)$. Hence, $a = 6$.

Now, $a^2 = b^2 - b^2e^2$

$\Rightarrow \quad b^2 = 36 + 144 = 180$

$\Rightarrow \quad e = \sqrt{1 - \dfrac{36}{180}} = \sqrt{1 - \dfrac{1}{5}} = \dfrac{2}{\sqrt{5}}$

29. A $\rightarrow$ p, q; B $\rightarrow$ r, s; C $\rightarrow$ p; D $\rightarrow$ p,q

A. Let $I = \int \dfrac{2^x}{\sqrt{1 - 4^x}}\, dx$

Put $2^x = t$, $2^x \log 2\, dx = dt$

$$\therefore \quad I = \frac{1}{\log 2}\int \frac{1}{\sqrt{1-t^2}}\,dt = \frac{1}{\log 2}\sin^{-1}\left(\frac{t}{1}\right) + C$$

$$= \frac{1}{\log 2}\sin^{-1}(2^x) + C$$

$$\therefore \quad k = \frac{1}{\log 2}$$

B. Let $I = \int \frac{dx}{(\sqrt{x})^2 + (\sqrt{x})^7} = \int \frac{dx}{(\sqrt{x})^7\left(1 + \frac{1}{(\sqrt{x})^5}\right)}$

Put $\frac{1}{(\sqrt{x})^5} = y \Rightarrow \frac{dy}{dx} = -\frac{5}{2(\sqrt{x})^7}$

$$\therefore \quad I = \int \frac{-2dy}{5(1+y)} = -\frac{2}{5}\ln|1+y| + C$$

$$= \frac{2}{5}\ln\left(\frac{1}{1 + \frac{1}{(\sqrt{x})^5}}\right)$$

$$\therefore \quad a = \frac{2}{5}, k = \frac{5}{2}$$

C. Add and subtract $2x^2$ in the numerator.
Then, $k = 1$ and $m = 1$.

D. Let $I = \int \frac{dx}{5 + 4\cos x}$

$$= \int \frac{dx}{5\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{9\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}}{9 + \tan^2\frac{x}{2}}\,dx$$

Put $t = \tan\frac{x}{2} \Rightarrow 2dt = \sec^2\frac{x}{2}\,dx$

$$\therefore \quad I = \int \frac{2dt}{9+t^2} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{3}\right) + C$$

$$\therefore \quad k = \frac{2}{3}, m = \frac{1}{3}$$

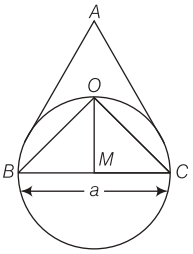
30. A → p,q; B → r,s; C → p,q; D → p,q

When O is circumcentre, then in ΔOMB, $OM = R\cos A$

[distance of circumcentre from BC]

Similarly, distance of circumcentre from AC and AB are $R\cos B$ and $R\cos C$.

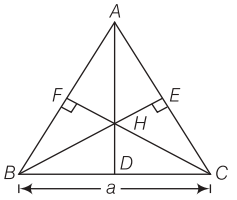
Applying sine rule in ΔOBC, we have



$$2R_1 = \frac{a}{\sin 2A} = \frac{a}{2\sin A\cos A} = \frac{R}{\cos A} \Rightarrow R_1 = \frac{R}{2\cos A}$$

Similarly, $R_2 = \frac{R}{\cos B}$ and $R_3 = \frac{R}{\cos C}$

[where, R_1, R_2, R_3 are circumradii of ΔOBC, ΔOCA and ΔOAB, respectively]



In ΔAFH, $AH = \frac{AF}{\cos\angle FAH}$

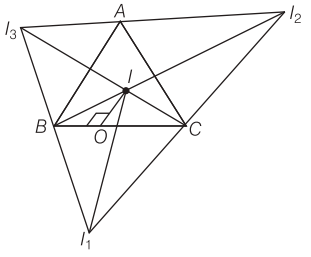
$$= \frac{b\cos A}{\cos(90^\circ - B)}$$

$$= 2R\cos A$$

Similarly, $BH = 2R\cos B$, $CH = 2R\cos C$

Also, $HF = AF\tan\angle FAH = b\cos A\cot B = 2R\cos A\cos B$

Similarly, $HE = 2R\cos A\cos C$ and $HD = 2R\cos B\cos C$



In ΔIDB, $BI = \frac{ID}{\sin\angle IBD} = \frac{r}{\sin(B/2)}$

Similarly, $CI = \frac{r}{\sin(C/2)}$

and $AI = \frac{r}{\sin(A/2)}$

Also, in ΔIBI_1,

$$II_1 = \frac{BI}{\cos\angle BII_1} = \frac{r/[\sin(B/2)]}{\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)}$$

$$= \frac{r}{\sin(B/2)\sin(C/2)}$$

Similarly, $II_2 = \frac{r}{\sin(A/2)\sin(B/2)}$

and $II_3 = \frac{r}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}$

Quick Fact

The equal sign, "=", was invented in 1557 by welsh mathematician Robert Recorde, who was fed up with writting "is equal to" in his equations. He choose the two lines because "no two things can be more equal".

Problem Solving Strategies On PROPERTIES OF TRIANGLES

Here, we are introducing some important strategies on Properties of Triangles to save your precious time in tedious calculations based problems asked in various Engineering Entrance exams. Use these strategies to practice questions and be acquainted with it.

1 In a ΔABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b, c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then [IIT JEE 2009]

(a) $b + c = 4a$
(b) $b + c = 2a$
(c) locus of point A is an ellipse
(d) locus of point A is a pair of straight lines

STRATEGY If $\cos B + \cos C = 4\sin^2\frac{A}{2}$, then $\sin B + \sin C = 2\sin A$.

Sol. ***(b,c)*** We have,

$$\cos B + \cos C = 4\sin^2\frac{A}{2}$$
$$\Rightarrow \quad \sin B + \sin C = 2\sin A$$
$$\Rightarrow \quad b + c = 2a$$
$$\Rightarrow \quad AC + AB = 2BC$$

$\therefore$ Locus of point A is an ellipse having its foci at B and C.

2 If in a ΔABC, $\sin A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$, then $\sin C$ is equal to [IIT JEE 2004]

(a) $\frac{33}{65}$ *(b)* $\frac{56}{65}$ *(c)* $\frac{33}{56}$ *(d)* None of these

STRATEGY Use $\tan A \tan B \tan C = \tan A + \tan B + \tan C$

Sol. ***(b)*** We have, $\sin A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$

$$\Rightarrow \quad \tan A = \frac{4}{3} \text{ and } \tan B = \frac{12}{5}$$

$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \quad \frac{4}{3} + \frac{12}{5} + \tan C = \frac{4}{3} \times \frac{12}{5} \times \tan C$$

$$\Rightarrow \quad \tan C = \frac{56}{33} \Rightarrow \sin C = \frac{56}{65}$$

3 In a ΔABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle, then $2(r + R)$ is equal to [IIT JEE 2002]

(a) $a + b$ *(b)* $b + c$ *(c)* $c + a$ *(d)* $a + b + c$

STRATEGY In a right angled triangle, the circumradius is half of hypotenuse.

Sol. ***(a)*** We have,

$$R = \frac{c}{2} \text{ and } r = (s - c)\tan\frac{C}{2} = s - c$$
$$\therefore \quad 2(R + r) = 2\left(\frac{c}{2} + s - c\right) = 2s - c = a + b$$

4 If Δ denotes the area of any triangle with semi-perimeter s, then [WB JEE 2007]

(a) $\Delta < \frac{s^2}{2}$ *(b)* $\Delta > \frac{s^2}{4}$ *(c)* $\Delta < \frac{s^2}{4}$ *(d)* $\Delta < s^2$

STRATEGY AM > GM

Sol. ***(c)*** Let a, b, c be the lengths of the sides of the triangle.

Now, AM > GM

$$\frac{s + (s - a) + (s - b) + (s - c)}{4} > [s(s - a)(s - b)(s - c)]^{\frac{1}{4}}$$
$$\Rightarrow \quad \frac{s}{2} > \Delta^{2/4}$$
$$\Rightarrow \quad \Delta < \frac{s^2}{4}$$

5 In a ΔABC, if $r_1 = 36$, $r_2 = 18$ and $r_3 = 12$, then the perimeter of the triangle is [EAMCET 2009]

(a) 36 *(b)* 18
(c) 72 *(d)* None of these

STRATEGY Use the result $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ and $\Delta^2 = rr_1r_2r_3$.

Sol. ***(c)*** We know that, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{36} + \frac{1}{18} + \frac{1}{12}$$

$$\Rightarrow \quad r = 6$$

Also, $\Delta^2 = r r_1 r_2 r_3$

$$= 6 \times 36 \times 18 \times 12$$

$$\Rightarrow \quad \Delta = 216$$

$$\therefore \quad s = \frac{\Delta}{r} = \frac{216}{6} = 36$$

$\therefore$ Perimeter $= 2s = 72$

6 In a ΔABC, the lengths of the two larger sides are 10 and 9 units, respectively. If the angles are in AP, then the lengths of the third side can be **[IIT JEE 2001]**

(a) $5 \pm \sqrt{6}$ (b) $3\sqrt{3}$ (c) 5 (d) $5\sqrt{3}$

STRATEGY If angles A, B and C of ΔABC are in AP, then $B = 60°$.

Sol. **(a)** Since, A, B, C are in AP, then $B = 60°$

Let $a = 10$ and $b = 9$

Now, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$$\Rightarrow \quad \frac{1}{2} = \frac{100 + c^2 - 81}{2 \times 10 \times c} \Rightarrow c^2 - 10c + 19 = 0$$

$$\Rightarrow \quad c = 5 \pm \sqrt{6}$$

7 In a ΔABC, the length of the median AD to the side BC is 4 units. If $\angle A = 60°$ and the area of the triangle is $2\sqrt{3}$ sq units, then the length of side BC is **[IIT JEE 2003]**

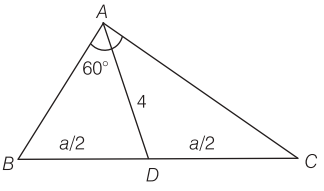
(a) $2\sqrt{3}$ (b) $4\sqrt{3}$ (c) 6 (d) 8

STRATEGY In a ΔABC, if AD is a median, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$.

Sol. **(b)** We have, $\angle A = 60°$

$$\Rightarrow \quad \cos A = \frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \quad b^2 + c^2 - a^2 = bc \qquad \ldots(i)$$



It is given that,

$$\Delta = 2\sqrt{3}$$

$$\Rightarrow \quad \frac{1}{2} bc \sin A = 2\sqrt{3}$$

$$\Rightarrow \quad \frac{1}{2} bc \sin 60° = 2\sqrt{3}$$

$$\Rightarrow \quad bc = 8 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$b^2 + c^2 - a^2 = 8 \Rightarrow b^2 + c^2 = a^2 + 8 \qquad \ldots(iii)$$

In ΔABC, we get

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

$$\Rightarrow \quad c^2 + b^2 = 2\left(16 + \frac{a^2}{4}\right)$$

$$\Rightarrow \quad 2(b^2 + c^2) = 64 + a^2 \qquad \ldots(iv)$$

From Eqs. (iii) and (iv), we get

$$2(a^2 + 8) = 64 + a^2 \Rightarrow a^2 = 48$$

$$\Rightarrow \quad a = 4\sqrt{3}$$

8 In a triangle, if $r_1 = 2r_2 = 3r_3$, then $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a}$ is equal to **[EAMCET 2008]**

(a) $\dfrac{75}{60}$ (b) $\dfrac{155}{60}$ (c) $\dfrac{176}{60}$ (d) $\dfrac{191}{60}$

STRATEGY If $r_1 = 2r_2 = 3r_3$, then $a : b : c = 5 : 4 : 3$.

Sol. **(d)** $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} = \dfrac{5}{4} + \dfrac{4}{3} + \dfrac{3}{5} = \dfrac{191}{60}$

9 In ΔABC, if a, b, c are in AP and $A - C = 90°$, then $\sin B$ equals to **[WB JEE 2002]**

(a) $\dfrac{\sqrt{7}}{5}$ (b) $\dfrac{\sqrt{5}}{8}$ (c) $\dfrac{\sqrt{7}}{4}$ (d) $\dfrac{\sqrt{5}}{3}$

STRATEGY If a, b, c are in AP, then $\sin\dfrac{B}{2} = \dfrac{1}{2}\cos\dfrac{A-C}{2}$.

Sol. **(c)** We have, a, b, c are in AP.

$$\Rightarrow \quad \sin\frac{B}{2} = \frac{1}{2}\cos\frac{A-C}{2}$$

$$\Rightarrow \quad \sin\frac{B}{2} = \frac{1}{2}\cos 45° \qquad [\because A - C = 90°]$$

$$\Rightarrow \quad \sin\frac{B}{2} = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \quad \cos\frac{B}{2} = \frac{\sqrt{7}}{2\sqrt{2}}$$

$$\therefore \quad \sin B = 2\sin\frac{B}{2}\cos\frac{B}{2} = \frac{\sqrt{7}}{4}$$

10 In a ΔABC, if the sides a, b, c are roots of $x^3 - 11x^2 + 38x - 40 = 0$, then the value of $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c}$ is equal to **[EAMCET 2005]**

(a) $\dfrac{9}{16}$ (b) $\dfrac{16}{9}$ (c) $\dfrac{9}{17}$ (d) $\dfrac{17}{9}$

STRATEGY Use the properties of roots of a cubic equation.

Sol. **(a)** Here, a, b and c are roots of equation

$$x^3 - 11x^2 + 38x - 40 = 0.$$

Therefore,

$$a + b + c = 11, ab + bc + ac = 38 \text{ and } abc = 40$$

$$\therefore \quad \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

$$= \frac{(a + b + c)^2 - 2(ab + bc + ca)}{2abc}$$

$$= \frac{11^2 - 76}{80}$$

$$= \frac{45}{80} = \frac{9}{16}$$

FÜN ARCADE

October 2016 issue Answers

Crossword

1H	Y	2P	E	R	B	O	3L	A										
		A					O				4R							
		R					5C	I	R	C	L	E			6L			
		A					A					A			I			
		L					L					L			M			
		L		7F			8M	A	T	R	I	X			A			
		E		A			A		9R				10C		C			
		L		C			X		E				O		O			
11P	O	I	N	T	O	F	I	N	F	L	E	C	T	I	O	N		
				O			M		L				A					
				R			A		E				N					
				I					X				G					
				A					I				12E	L	L	I	13P	S	E
				L					V				N				O		
		14U							E		15I	O	T	A			L		
	16I	N	T	E	G	R	A	T	I	N	G						A		
		I															R		
		17T	A	N	G	E	N	T											

Puzzles to Puzzle you

1. 70, 72, 74, 80
2. 139854276, 923187456
3. 6, 4.7, 7; 264, 198, 308
4. 5 **5.** 187 **6.** 4, 5, 6

SUDOKU

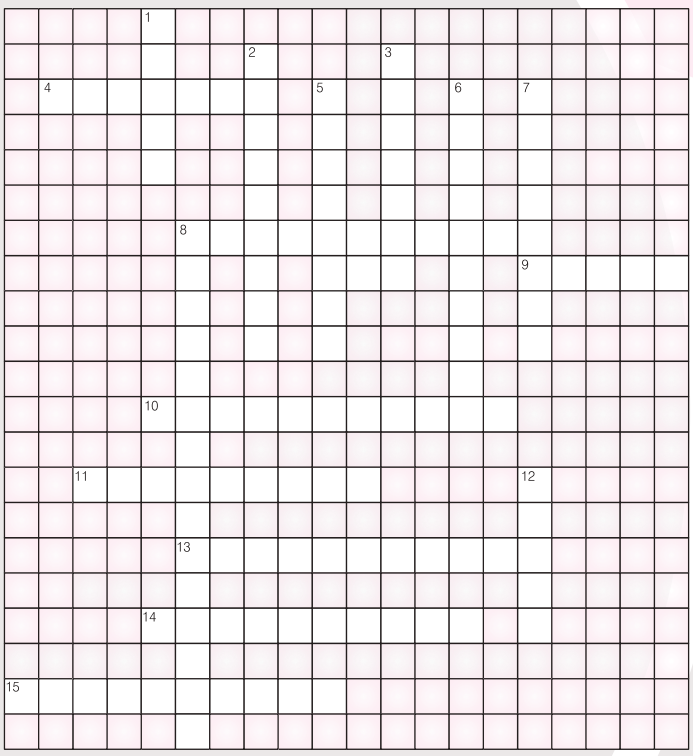
8	1	6	7	2	4	9	3	5
2	9	3	6	5	8	7	4	1
4	7	5	9	1	3	2	6	8
9	3	8	2	6	5	4	1	7
5	4	1	8	7	9	3	2	6
6	2	7	3	4	1	5	8	9
1	5	9	4	8	2	6	7	3
3	6	2	1	9	7	8	5	4
7	8	4	5	3	6	1	9	2

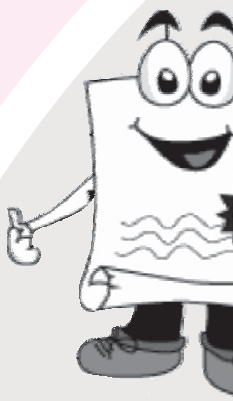
DOWN

1. Another name of gradient. (5)
2. A complex number $(x+iy)$ can always be rewritten in the form $r(\cos\theta + i\sin\theta)$, in which symbol is the. (9)
3. The curve traced out by a point on a wheel rolling along a straight line. (7)
5. The intersecting point of three medians. (8)
6. Points or lines that all lie in the same plane. (8)
7. The perpendicular distance from one side to the farthest point. (8)
8. The process for finding a derivative. (15)
12. The set of all possible values for the output of the function. (5)

ACROSS

4. A general conclusion proposed to be proved upon the basis of certain given hypothesis or assumptions. (7)
8. A single value obtained by multiplying and adding the elements of a square matrix together. (11)
9. The set formed by combining all the elements of two or more sets. (5)
10. A sequence whose terms follow certain rule. (11)
11. An experiment in which there are two possible independent results then trials are known as ……trials. (9)
13. Another name of anti-derivative. (11)
14. The branch of mathematics that deals with the collection, organization, analysis and interpretation of numerical data. (10)
15. A curved, three-dimensional surface or solid formed by a parabola about its axis. (10)





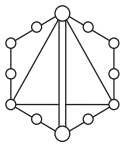
PUZZLES TO PUZZLE YOU

1. A plane has a span of 12 m. It was photographed as it was flying directly overhead with a camera with a depth of 12 cm. In the photo the span of the plane was 8 mm. Can you tell how high was the plane when it was snapped?

2. I found a very nice hand bag in the shop that I thought I should simply must have. The price was ₹ 10. But I did not have ₹10.
The storekeeper was known to me. He said that he had a credit system by which I could pay ₹ 1 only at the time of purchase and the balance ₹ 10 could be paid by me at the rate of ₹ 1 per week for 10 weeks.
Can you tell what annual rate of interest the storekeeper was charging me?

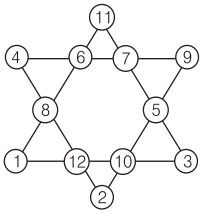
3. A farmer has six baskets full of chicken and duck eggs he is taking to the market to sell. Basket A has 29 eggs, basket B has 14 eggs, basket C has 23 eggs, basket D has 6 eggs, basket E has 5 eggs and basket F has 12 eggs.
The farmer points to a particular basket and says "If I sell the eggs in this basket, I shall have twice as many chicken eggs left as duck eggs".
Which was the basket he was pointing at?

4. Can you arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 in the twelve circles of their figure, so that the sum of the three numbers an each of the six sides is the same and the three circles connected by single line and the two circles connected by the double line also give the same sum?



5. Take a good look at this six-painted figure. This is what is known as a magic star. The total in every row adds upto the same.

$1 + 12 + 10 + 3 = 26$
$4 + 6 + 7 + 9 = 26$
$4 + 8 + 12 + 2 = 26$
$9 + 5 + 10 + 2 = 26$
$11 + 6 + 8 + 1 = 26$
$11 + 7 + 5 + 3 = 26$



But there is something imperfect about this star. The sum of the numbers at the points do not add upto 26. Now, can you replace the numbers in such a way that their sums in every row and every point add upto 26?

6. My friend Parveen teaches at a school. One day she conducted a test for three of her students and when they handed back the test papers, they had forgotten to write their names. Parveen returned the papers to the students at random. What is the probability that name of the three students will get the right paper?

SUDOKU

	6					3		
	2	4	6	7				
			8		9			
			9			5		
					1			6
2	1		5					3
					3			7
1				2		9		
8								

Personality Spectrum

Leonhard Euler (1707-1783)

Leonhard Euler was an 18th century physicist and scholar, who was responsible for developing many concepts that are an integral part of modern Mathematics.
Born on 15th April, 1707, in Basel , Switzerland, he was one of math's most pioneering thinkers, establishing a career as an academy scholar and contributing greatly to the fields of geometry, trigonometry and calculus, among many others . He released hundred of articles and publications during his lifetime and continued to publish after losing his sight. He died on 18th September, 1783.
After studying with Johan Bernoulli, he attended the University of Basel and earned his master's during his teens. Moving to Russia in 1727, Euler served in the navy before joining the St. Petersburg Academy as a professor of Mathematics.
In 1736, he published his first book of many, Mechanica. By the end of the decade, having suffered from fever and overexertion due to cartography work, Euler was severly hampered in the ability to see from his right eye.

3-D, LIMITS AND DERIVATIVES

FOR CLASS XI

Topicwise Collection of Best Subjective Problems

Very Short Answer Type Questions [1 Mark]

1. Write the distance of the point $P(2, 3, 5)$ from the XY-plane.

2. Find the coordinates of a point equidistant from the four points $O(0, 0, 0)$, $A(l, 0, 0)$, $B(0, m, 0)$ and $C(0, 0, n)$. [NCERT Exemplar]

3. What is the locus of a point (x, y, z) for which $y = z = 0$?

4. If the extremities of the diagonals of a square are $(1, -2, 3)$ and $(2, -3, 5)$, then find the length of the side.

5. If a parallelopiped is formed by planes drawn through the points $(2, 3, 5)$ and $(5, 9, 7)$ parallel to the coordinate planes, then find the length of edges of the parallelopiped. [NCERT Exemplar]

6. Write the value of $\lim_{x \to 0^+} [x]$.

7. Let $f(x) = \begin{cases} \cos x, & \text{if } x > 0 \\ x + k, & \text{if } x < 0 \end{cases}$. Find the value of constant k, given that $\lim_{x \to 0} f(x)$ exists.

8. If $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$ and $n \in N$, find n. [NCERT Exemplar]

9. Write the value of $\lim_{n \to \infty} \frac{n! + (n+1)!}{(n+1)! + (n+2)!}$.

10. Evaluate $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$. [HOTS]

11. If $f(x) = \alpha x^n$, then prove that $\alpha = \frac{f'(1)}{n}$.

12. If $f(x) = \log_{x^2} x^3$, then write the value of $f'(x)$.

13. If $\frac{\pi}{2} < x < \pi$, then find $\frac{d}{dx}\left(\sqrt{\frac{1 + \cos 2x}{2}}\right)$.

14. If $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$, then find $f'(1)$. [NCERT Exemplar]

15. Differentiate $x \sin x \log x$ with respect to x.

Short Answer Type Questions [4 Marks]

16. Determine the point in XY-plane which is equidistant from three points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.

17. Find the ratio in which the line joining the points $A(2, 1, 5)$ and $B(3, 4, 3)$ is divided by the plane $2x + 2y - 2z = 1$. Also, find the coordinates of the point of division. [HOTS]

18. Show that the coordinates of the centroid of the triangle with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

19. If $A(-2, 2, 3)$ and $B(13, -3, 13)$ are two points, find the locus of a point P which moves in such a way that $3PA = 2PB$.

20. If f is an odd function and if $\lim_{x \to 0} f(x)$ exists. Prove that this limit must be zero. [HOTS]

21. Let $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{where } x \neq \pi/2 \\ 3, & \text{where } x = \pi/2 \end{cases}$ and if $\lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$, then find the value of k. [NCERT Exemplar]

22. Evaluate $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$. [NCERT Exemplar]

23. Evaluate $\lim_{x \to e} \frac{\log x - 1}{x - e}$. [HOTS]

To be Continued at Page 78

TOUCH UP

VECTOR AND PROBABILITY

FOR CLASS XII

Topicwise Collection of Best Subjective Problems

Very Short Answer Type Questions [1 Mark]

1. In a ΔOAC, if B is the mid-point of side AC and $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, then what is $\mathbf{OC}$? [AI 2015]

2. Find a vector in the direction of vector $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ which has magnitude 21 units. [AI 2014]

3. Find a vector $\mathbf{a}$ of magnitude $5\sqrt{2}$ making an angle of $\frac{\pi}{4}$ with X-axis, $\frac{\pi}{2}$ with Y-axis and an acute angle θ with Z-axis. [AI 2014]

4. Find the value of 'p' for which the vectors $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 2p\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ are parallel. [AI 2014]

5. Write the direction cosines of the vectors $-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$. [AI 2011]

6. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then find the angle between $\mathbf{a}$ and $\mathbf{b}$, given that $(\sqrt{3}\mathbf{a} - \mathbf{b})$ is a unit vector. [AI 2014]

7. Write the projection of $\mathbf{b} + \mathbf{c}$ on $\mathbf{a}$, when $\mathbf{a} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. [AI 2013]

8. Write the value of $(\mathbf{a}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{a}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{a}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}$, where $\mathbf{a}$ is any vector.

9. Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. [AI 2015]

10. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then write the value of $|\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2$.

11. Write the value of $\hat{\mathbf{i}}\times(\hat{\mathbf{j}} + \hat{\mathbf{k}}) + \hat{\mathbf{j}}\times(\hat{\mathbf{k}} + \hat{\mathbf{i}}) + \hat{\mathbf{k}}\times(\hat{\mathbf{i}} + \hat{\mathbf{j}})$. [AI 2012]

12. Find $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, if $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. [AI 2014]

13. If $\mathbf{a}$ and $\mathbf{b}$ are non-collinear vectors, then find the value of $[\mathbf{a}\ \mathbf{b}\ \hat{\mathbf{i}}]\hat{\mathbf{i}} + [\mathbf{a}\ \mathbf{b}\ \hat{\mathbf{j}}]\hat{\mathbf{j}} + [\mathbf{a}\ \mathbf{b}\ \hat{\mathbf{k}}]\hat{\mathbf{k}}$.

14. Evaluate $[\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}] + [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}]$.

15. Find the volume of the parallelopiped with its edges represented by the vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \pi\hat{\mathbf{k}}$.

Short Answer Type Questions [4 Marks]

16. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors inclined at an angle θ, then prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\mathbf{a} - \mathbf{b}|$. [NCERT Exemplar]

17. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ are given vectors, then find vector $\mathbf{b}$ satisfying the equations $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ and $\mathbf{a}\cdot\mathbf{b} = 3$. [NCERT Exemplar, AI 2013]

18. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors such that $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = 0$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\pi/6$, prove that $\mathbf{a} = \pm 2(\mathbf{b}\times\mathbf{c})$.

19. For any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, prove that $[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$. [AI 2014]

20. If the vectors $\vec{\alpha} = a\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\vec{\beta} = \hat{\mathbf{i}} + b\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\gamma} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + c\hat{\mathbf{k}}$ are coplanar, then prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$, where $a \neq 1$, $b \neq 1$ and $c \neq 1$.

21. If A and B are two independent events such that $P(\overline{A} \cap B) = \frac{2}{15}$ and $P(A \cap \overline{B}) = \frac{1}{6}$, then find $P(A)$ and $P(B)$. [AI 2015]

22. A bag contains $(2n + 1)$ coins. It is known that n of these coins have a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss result in a head is $\frac{31}{42}$, determine the value of n. [NCERT Exemplar]

23. The probability distribution of a random variable X is given below

X	0	1	2	3	4
$P(X)$	0.1	0.25	0.3	0.2	0.15

Find (i) Var (X) (ii) Var $\left(\frac{X}{2}\right)$. [NCERT Exemplar]

24. For 6 trials of an experiment, let X be a binomial variate which satisfies the relation $9P(X = 4) = P(X = 2)$. Find the probability of success. [AI 2015]

25. Three dice are thrown at the same time find the probability of getting three two's, if it is known that the sum of the numbers on the dice was a six.

Long Answer Type Questions [6 Marks]

26. A bag contains n white and n red balls. Pair of balls are drawn without replacement untill the bag is empty. Show that the probability that each pair consists of one white and one red ball is $\frac{2^n}{{}^{2n}C_n}$. [HOTS]

27. A and B throw alternatively a pair of dice. A wins, if he throws 6 before B throws 7 and B wins, if he throws 7 before A throws 6. Find their respective chance of winning, if A begins. [NCERT Exemplar]

28. Let d_1, d_2, d_3 be three mutually exclusive diseases. Let $S = \{s_1, s_2, s_3, \ldots, s_6\}$ be the set of observable symptoms of these diseases. e.g. s_1 is the shortness of breath, s_2 is loss of weight, s_3 is fatigue etc. Suppose a random sample of 10000 patients contains 3200 patients with disease d_1, 3500 with disease d_2 and 3300 with disease d_3. Also, 3100 patients with disease d_1, 3300 with disease d_2 and 3000 with disease d_3 show the symptom S. Knowing that the patient has symptom S, the doctor wishes to determine the patient's illness. On the basis of this information, what should the doctor conclude? [VBQ]

29. A shopkeeper sells three types of seeds A_1, A_2 and A_3. They are sold as a mixture where the proportions are 4 : 4 : 2, respectively. The germination rates of three types of seeds are 45%, 60% and 35% Calculate the probability

(i) that it will not germinate given that the seed is of type A_3.

(ii) of a randomly chosen seed to germinate.

(iii) that it is of type A_2 given that a randomly chosen seed does not germinate. [NCERT Exemplar]

30. Let X be a discrete random variable whose probability distribution is defined as follows

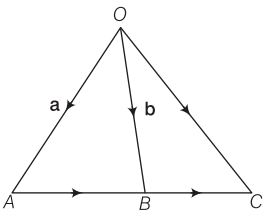
$$P(X = x) = \begin{cases} k(x+1), & \text{for } x = 1, 2, 3, 4 \\ 2kx, & \text{for } x = 5, 6, 7 \\ 0, & \text{otherwise} \end{cases}$$

where, k is a constant.

Find (i) k (ii) $E(X)$ (iii) $\sigma(X)$.

Solutions

1.



In ΔOAB,

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{b} - \mathbf{a}$$

Since, B is a mid-point of AC.

$$\therefore \quad \mathbf{AC} = 2\mathbf{AB} = 2\mathbf{b} - 2\mathbf{a}$$

Now, in ΔOAC,

$$\mathbf{OC} = \mathbf{OA} + \mathbf{AC}$$
$$= \mathbf{a} + 2\mathbf{b} - 2\mathbf{a} = 2\mathbf{b} - \mathbf{a}$$

2. Let $\mathbf{a} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$. Then, $|\mathbf{a}| = \sqrt{(2)^2 + (-3)^2 + 6^2} = 7$

$\therefore$ A unit vector parallel to $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$

$$\Rightarrow \quad \hat{\mathbf{a}} = \frac{1}{7}(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = \frac{2}{7}\hat{\mathbf{i}} - \frac{3}{7}\hat{\mathbf{j}} + \frac{6}{7}\hat{\mathbf{k}}$$

Hence, required vector = $21\ \hat{\mathbf{a}}$

$$= 21\left(\frac{2}{7}\hat{\mathbf{i}} - \frac{3}{7}\hat{\mathbf{j}} + \frac{6}{7}\hat{\mathbf{k}}\right) = 6\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + 18\hat{\mathbf{k}}$$

3. Let l, m, n be the direction cosines of $\mathbf{a}$.

It is given that $\mathbf{a}$ is inclined at $\frac{\pi}{4}$ to OX and $\frac{\pi}{2}$ to OY.

$$\therefore \quad l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } m = \cos\frac{\pi}{2} = 0$$

But $\quad l^2 + m^2 + n^2 = 1$

$$\therefore \quad \frac{1}{2} + 0 + n^2 = 1$$

$$\Rightarrow \quad n^2 = \frac{1}{2} \Rightarrow n = \frac{1}{\sqrt{2}}$$

Thus, we have

$$l = \frac{1}{\sqrt{2}},\ m = 0,\ n = \frac{1}{\sqrt{2}} \text{ and } |\mathbf{OA}| = 5\sqrt{2}$$

$\therefore \quad \mathbf{OA} = |\mathbf{OA}|(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})$

$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \frac{\hat{\mathbf{k}}}{\sqrt{2}}\right) = 5\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

4. Since, the vectors $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 2p\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ are parallel.

$\therefore \quad \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3} \qquad \left[\because \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\right]$

$\Rightarrow \quad p = -\frac{1}{3}$

5. Direction cosines of the given vector are

$$\frac{-2}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}, \frac{1}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}, \frac{5}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}$$

$$= \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}}$$

6. We have, $\sqrt{3}\mathbf{a} - \mathbf{b}$ is a unit vector.

$\therefore \quad |\sqrt{3}\mathbf{a} - \mathbf{b}| = 1 \Rightarrow |\sqrt{3}\mathbf{a} - \mathbf{b}|^2 = 1$

$\Rightarrow \quad (\sqrt{3}\mathbf{a} - \mathbf{b}) \cdot (\sqrt{3}\mathbf{a} - \mathbf{b}) = 1$

$\Rightarrow \quad 3\mathbf{a} \cdot \mathbf{a} - \sqrt{3}\,\mathbf{a} \cdot \mathbf{b} - \sqrt{3}\,\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = 1$

$\Rightarrow \quad 3|\mathbf{a}|^2 - 2\sqrt{3}|\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta + |\,\mathbf{b}\,|^2 = 1$

[$\because \theta$ is angle between $\mathbf{a}$ and $\mathbf{b}$]

$\Rightarrow \quad 3 - 2\sqrt{3}\cos\theta + 1 = 1 \qquad [\because |\,\mathbf{a}\,| = |\,\mathbf{b}\,| = 1]$

$\Rightarrow \quad \cos\theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$

7. We have,

$\mathbf{a} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

and $\mathbf{c} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\therefore \quad \mathbf{b} + \mathbf{c} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) + (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\therefore$ Projection of $(\mathbf{b} + \mathbf{c})$ on $\mathbf{a} = \frac{\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})}{|\mathbf{a}|}$

$$= \frac{(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{|2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}|}$$

$$= \frac{6 - 2 + 2}{|4 + 4 + 1|} = \frac{6}{3} = 2$$

8. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

$\therefore \quad \mathbf{a} \cdot \hat{\mathbf{i}} = a_1, \mathbf{a} \cdot \hat{\mathbf{j}} = a_2, \mathbf{a} \cdot \hat{\mathbf{k}} = a_3$

$\therefore \quad (\mathbf{a} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{a} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{a} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} = \mathbf{a}$

9. We have, $\mathbf{a} = i + 2j - 3k$ and $\mathbf{b} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Then, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -3 \\ 3 & -1 & 2 \end{vmatrix}$

$= (4 - 3)\hat{\mathbf{i}} - (2 + 9)\hat{\mathbf{j}} + (-1 - 6)\hat{\mathbf{k}}$

$= \hat{\mathbf{i}} - 11\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$

$\Rightarrow \quad |\,\mathbf{a} \times \mathbf{b}\,| = \sqrt{(1)^2 + (-11)^2 + (-7)^2}$

$= \sqrt{171}$

$\therefore$ Required vector $= \sqrt{171}\left\{\frac{\mathbf{a} \times \mathbf{b}}{|\,\mathbf{a} \times \mathbf{b}\,|}\right\}$

$= \frac{\sqrt{171}}{\sqrt{171}}(\hat{\mathbf{i}} - 11\hat{\mathbf{j}} - 7\hat{\mathbf{k}})$

$= \hat{\mathbf{i}} - 11\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$

10. Let θ be the angle between the unit vectors $\mathbf{a}$ and $\mathbf{b}$.

$\therefore \quad |\,\mathbf{a} \times \mathbf{b}\,|^2 + (\mathbf{a} \cdot \mathbf{b})^2$

$= (|\,\mathbf{a}\,||\,\mathbf{b}\,|\sin\theta)^2 + (|\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta)^2$

$= \sin^2\theta + \cos^2\theta \qquad [\because |\mathbf{a}| = |\mathbf{b}| = 1]$

$= 1$

11. $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} + \hat{\mathbf{k}}) + \hat{\mathbf{j}} \times (\hat{\mathbf{k}} + \hat{\mathbf{i}}) + \hat{\mathbf{k}} \times (\hat{\mathbf{i}} + \hat{\mathbf{j}})$

$= \hat{\mathbf{i}} \times \hat{\mathbf{j}} + \hat{\mathbf{i}} \times \hat{\mathbf{k}} + \hat{\mathbf{j}} \times \hat{\mathbf{k}} + \hat{\mathbf{j}} \times \hat{\mathbf{i}} + \hat{\mathbf{k}} \times \hat{\mathbf{i}} + \hat{\mathbf{k}} \times \hat{\mathbf{j}}$

$= \hat{\mathbf{k}} - \hat{\mathbf{j}} + \hat{\mathbf{i}} - \hat{\mathbf{k}} + \hat{\mathbf{j}} - \hat{\mathbf{i}}$

$= 0$

12. We have,

$\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\therefore \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 1 & 2 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

$= 2(4 - 1) - 1(-2 - 3) + 2(-1 - 6)$

$= 6 + 5 - 14$

$= -3$

13. For any vector $\mathbf{r}$, we have

$\mathbf{r} = (\mathbf{r} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{r} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{r} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$

Replacing $\mathbf{r}$ by $\mathbf{a} \times \mathbf{b}$, we have

$\mathbf{a} \times \mathbf{b} = \{(\mathbf{a} \times \mathbf{b}) \cdot \hat{\mathbf{i}}\}\hat{\mathbf{i}} + \{(\mathbf{a} \times \mathbf{b}) \cdot \hat{\mathbf{j}}\}\hat{\mathbf{j}} + \{(\mathbf{a} \times \mathbf{b}) \cdot \hat{\mathbf{k}}\}\hat{\mathbf{k}}$

$= [\mathbf{a}\ \mathbf{b}\ \hat{\mathbf{i}}]\hat{\mathbf{i}} + [\mathbf{a}\ \mathbf{b}\ \hat{\mathbf{j}}]\hat{\mathbf{j}} + [\mathbf{a}\ \mathbf{b}\ \hat{\mathbf{k}}]\hat{\mathbf{k}}$

14. $[\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}] + [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}]$

$= (\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \cdot \hat{\mathbf{k}} + (\hat{\mathbf{i}} \times \hat{\mathbf{k}}) \cdot \hat{\mathbf{j}}$

$= \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} + (-\hat{\mathbf{j}}) \cdot \hat{\mathbf{j}}$

$= \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} - \hat{\mathbf{j}} \cdot \hat{\mathbf{j}}$

$= 1 - 1$

$= 0$

15. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}}, \mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}, \mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \pi\hat{\mathbf{k}}$

The volume of parallelopiped whose three adjacent edges are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is equal to $|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$.

$\therefore$ Volume $= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix}$

$= \pi(2 - 1)$

$= \pi$ cu units

16. We know that, $\mathbf{x}\cdot\mathbf{x} = |\mathbf{x}|^2$

$\therefore \quad |\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|^2 = \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \qquad [\because \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}]$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta + |\mathbf{b}|^2$

$[\because \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta]$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|^2 = 2 - 2\cos\theta \qquad [\because |\mathbf{a}| = |\mathbf{b}| = 1]$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|^2 = 2(1-\cos\theta) = 2\left(2\sin^2\frac{\theta}{2}\right) = 4\sin^2\frac{\theta}{2}$

$\Rightarrow \quad \sin^2\frac{\theta}{2} = \frac{1}{4}|\mathbf{a}-\mathbf{b}|^2$

$\Rightarrow \quad \sin\frac{\theta}{2} = \frac{1}{2}|\mathbf{a}-\mathbf{b}|$ **Hence proved.**

17. Let $\mathbf{b} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Then, $\mathbf{a}\times\mathbf{b} = \mathbf{c}$

$\Rightarrow \quad \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = (\hat{\mathbf{j}} - \hat{\mathbf{k}})$

$\Rightarrow \quad (z-y)\hat{\mathbf{i}} - (z-x)\hat{\mathbf{j}} + (y-x)\hat{\mathbf{k}} = 0\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$

$\Rightarrow \quad z - y = 0, -(z-x) = 1, y - x = -1$

$\Rightarrow \quad y = z, x - z = 1, x - y = 1$

$\Rightarrow \quad x - z = 1, x - y = 1$

$[\because$ these two equations are equivalent to $y = z]$

$\Rightarrow \quad z = x - 1$ and $y = x - 1 \qquad \ldots(i)$

It is given that, $\mathbf{a}\cdot\mathbf{b} = 3$

$\Rightarrow \quad (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 3$

$\Rightarrow \quad x + y + z = 3$

$\Rightarrow \quad x + x - 1 + x - 1 = 3 \qquad$ [using Eq. (i)]

$\Rightarrow \quad 3x = 5 \Rightarrow x = \frac{5}{3}$

$\therefore \quad y = x - 1 \Rightarrow y = \frac{5}{3} - 1 = \frac{2}{3}$

and $\quad y = z \Rightarrow z = \frac{2}{3}$

Hence, $\mathbf{b} = \frac{5}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}$

18. We have,

$\mathbf{a}\cdot\mathbf{b} = 0$ and $\mathbf{a}\cdot\mathbf{c} = 0$

$\Rightarrow \quad \mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$

$\Rightarrow \mathbf{a}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$.

$\Rightarrow \mathbf{a}$ is perpendicular to the plane containing vectors $\mathbf{b}$ and $\mathbf{c}$.

$\Rightarrow \quad \mathbf{a}$ is parallel to $\mathbf{b}\times\mathbf{c}$

$\Rightarrow \quad \mathbf{a} = \lambda(\mathbf{b}\times\mathbf{c}) \qquad \ldots(i)$

$\Rightarrow \quad |\mathbf{a}| = |\lambda||\mathbf{b}\times\mathbf{c}|$

$\Rightarrow \quad |\mathbf{a}| = |\lambda||\mathbf{b}||\mathbf{c}|\sin\frac{\pi}{6}$

$\Rightarrow \quad 1 = \frac{|\lambda|}{2} \qquad [\because |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1]$

$\Rightarrow \quad |\lambda| = 2$

$\Rightarrow \quad \lambda = \pm 2$

Putting this value of λ in Eq. (i), we get

$\mathbf{a} = \pm 2(\mathbf{b}\times\mathbf{c})$ **Hence proved.**

19. $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]$

$= \{(\mathbf{a}+\mathbf{b})\times(\mathbf{b}+\mathbf{c})\}\cdot(\mathbf{c}+\mathbf{a})$

$= (\mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{b} + \mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}+\mathbf{a})$

$= (\mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}+\mathbf{a}) \qquad [\because \mathbf{b}\times\mathbf{b} = 0]$

$= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} + (\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} + (\mathbf{a}\times\mathbf{c})\cdot\mathbf{c}$

$+ (\mathbf{a}\times\mathbf{c})\cdot\mathbf{a} + (\mathbf{b}\times\mathbf{c})\cdot\mathbf{c} + (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{a}] + [\mathbf{a}\ \mathbf{c}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{c}\ \mathbf{a}]$

$+ [\mathbf{b}\ \mathbf{c}\ \mathbf{c}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$ $[\because$ scalar triple product is zero when any two vectors are equal]

$= 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \qquad [\because [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]]$

Hence proved.

20. It is given that $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are coplanar vectors.

$\therefore \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0$

$\Rightarrow \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$\Rightarrow \quad abc - a - c + 1 + 1 - b = 0$

$\Rightarrow \quad abc = a + b + c - 2 \qquad \ldots(i)$

Now, $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$

$= \frac{(1-b)(1-c) + (1-c)(1-a) + (1-a)(1-b)}{(1-a)(1-b)(1-c)}$

$= \frac{3 - 2(a+b+c) + (ab+bc+ca)}{1 - (a+b+c) + (ab+bc+ca) - abc}$

$= \frac{3 - 2(a+b+c) + (ab+bc+ca)}{1 - (a+b+c) + (ab+bc+ca) - (a+b+c-2)}$

[using Eq. (i)]

$= \frac{3 - 2(a+b+c) + (ab+bc+ca)}{3 - 2(a+b+c) + ab + bc + ca}$

$= 1$ **Hence proved.**

21. Let $P(A) = x$ and $P(B) = y$. It is given that A and B are independent events such that

$P(\overline{A} \cap B) = \frac{2}{15}$ and $P(A \cap \overline{B}) = \frac{1}{6}$

$\Rightarrow \quad P(\overline{A})P(B) = \frac{2}{15}$ and $P(A)P(\overline{B}) = \frac{1}{6}$

$\Rightarrow \quad [1 - P(A)]P(B) = \frac{2}{15}$

and $\quad P(A)[1 - P(B)] = \frac{1}{6}$

$\Rightarrow \quad (1-x)y = \frac{2}{15}$ and $x(1-y) = \frac{1}{6}$

$\Rightarrow \quad y - xy = \frac{2}{15}$ ...(i)

and $\quad x - xy = \frac{1}{6}$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we obtain

$$x - y = \frac{1}{6} - \frac{2}{15} \Rightarrow x - y = \frac{1}{30}$$

$\Rightarrow \quad x = y + \frac{1}{30}$ ...(iii)

Putting $x = y + \frac{1}{30}$ in Eq. (i), we get

$$y - \left(y + \frac{1}{30}\right)y = \frac{2}{15}$$

$\Rightarrow \quad y - y^2 - \frac{1}{30}y = \frac{2}{15}$

$\Rightarrow \quad 30y^2 - 29y + 4 = 0$

$\Rightarrow \quad y = \frac{1}{6}, \frac{4}{5}$

When $y = \frac{1}{6}$, then $x = \frac{1}{5}$

and when $y = \frac{4}{5}$, then $x = \frac{5}{6}$

Thus, $\quad P(A) = \frac{1}{5}$ and $P(B) = \frac{1}{6}$

or $\quad P(A) = \frac{5}{6}$ and $P(B) = \frac{4}{5}$

22. A coin picked up at random from the bag may be a fair coin or a coin having head on both sides. The tossed coin may result in a head in the following mutually exclusive ways

(i) A fair coin is taken out of the bag and then it is tossed to get a head.

(ii) A two headed coin is taken out of the bag and then it is tossed to get a head.

So, let us define the following events

E_1 = Taking out a fair coin from the bag

E_2 = Taking out a coin having head on both sides

A = Tossed coin results in a head

Clearly, $P(E_1) = \frac{n+1}{2n+1}$ and $P(E_2) = \frac{n}{2n+1}$

Also, $P(A/E_1) = \frac{1}{2}$ and $P(A/E_2) = 1$

By total probability theorem,

$$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$$

$$= \frac{n+1}{2n+1} \times \frac{1}{2} + \frac{n}{2n+1} \times 1 = \frac{3n+1}{2(2n+1)}$$

But it is given that, $P(A) = \frac{31}{42}$

$\therefore \quad \frac{31}{42} = \frac{3n+1}{2(2n+1)}$

$\Rightarrow \quad 62n + 31 = 63n + 21$

$\Rightarrow \quad n = 10$

23. (i)

X_i	0	1	2	3	4
$P(X_i)$	0.1	0.25	0.3	0.2	0.15
$X_iP(X_i)$	0	0.25	0.6	0.6	0.6
$X_i^2P(X_i)$	0	0.25	1.2	1.8	2.40

Thus, we have $\Sigma P(X_i)X_i = 2.05$

and $\Sigma P(X_i)X_i^2 = 5.65$

$\therefore \quad \text{Var}(X) = \Sigma P(X_i)X_i^2 - (\Sigma P(X_i)X_i)^2$

$= 5.65 - (2.05)^2 = 1.4475$

(ii) We know that, $\text{Var}(aX + b) = a^2\,\text{Var}(X)$

$\therefore \quad \text{Var}\left(\frac{X}{2}\right) = \frac{1}{4}\text{Var}(X)$

$= \frac{1}{4} \times 1.4475 = 0.361875$

24. Let p be the probability of success and q that of failure in a single trial. Then,

$P(X = r) = {}^6C_r p^r q^{6-r}$ $[\because n = 6 \text{ (given)}]$

It is given that,

$9P(X = 4) = P(X = 2)$

$\Rightarrow \quad 9 \times {}^6C_4 p^4 q^2 = {}^6C_2 p^2 q^4$

$\Rightarrow \quad 9p^2 = q^2$ $[\because {}^6C_2 = {}^6C_4]$

$\Rightarrow \quad 3p = q$

$\Rightarrow \quad 3p = 1 - p$

$\Rightarrow \quad p = \frac{1}{4}$

25. Associated to the random experiment of throwing three dice there are $6 \times 6 \times 6 = 216$ elementary events. Consider the following events

A = Sum of the numbers on the dice is six

B = Getting three two's

We have to find $P(B/A)$

We observed that

$A = \{(1, 2, 3), (1, 3, 2), (2, 3, 1), (2, 1, 3), (3, 1, 2),$
$(3, 2, 1), (1, 1, 4), (1, 4, 1), (4, 1, 1), (2, 2, 2)\}$

and $B = \{(2, 2, 2)\}$

$\therefore P(A) = \frac{10}{216}$, $P(B) = \frac{1}{216}$ and $P(A \cap B) = \frac{1}{216}$

$\therefore$ Required probability $= P(B/A) = \frac{P(A \cap B)}{P(A)}$

$= \frac{1/216}{10/216} = \frac{1}{10} = 0.1$

Mint Twister

There is a 1000 gram watermelon laying out in the sun. 99% of the watermelon's weight is water. After laying out for a few hours 98% of the watermelon's weight is water. How much water evaporated.

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1. Find all positive real numbers x, y, z such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2014},$$
$$2y - 2z + \frac{1}{x} = \frac{1}{2014},$$
$$2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

2. Solve $(x+y)^{2/3} + 2(x-y)^{2/3} = 3(x^2 - y^2)^{1/3}$ and $3x - 2y = 13$

3. If $a + b + c = 1$, $a^2 + b^2 + c^2 = 9$ and $a^3 + b^3 + c^3 = 1$, then find $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

4. A student is allowed to select atmost n books from a collection of $(2n + 1)$ books. If the total number of ways in which he can select atleast one book is 63. Find the value of n.

5. Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

6. Prove that the equation

$$m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$$

represents three straight lines equally inclined to each other.

7. $ABCD$ is a parallelogram, X and Y are mid-points of BC and CD, respectively. Prove that

$$\text{ar}(\Delta AXY) = \frac{3}{8}\,\text{ar}(||\text{gm } ABCD).$$

8. If $f(x + y) = f(x) \cdot f(y)$, $\forall$ real x, y and $f(0) \neq 0$, then prove that the function $F(x) = \frac{f(x)}{1 + [f(x)]^2}$ is an even function.

9. Without evaluating square roots, find the larger of two surds

$$\sqrt{12} - \sqrt{11},\ \sqrt{7} - \sqrt{6}.$$

10. In ΔABC, let D be the mid-point of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

Solutions

1. We have, $2x - 2y + \frac{1}{z} = \frac{1}{2014}$ …(i)

$$2y - 2z + \frac{1}{x} = \frac{1}{2014} \quad \text{…(ii)}$$

$$2z - 2x + \frac{1}{y} = \frac{1}{2014} \quad \text{…(iii)}$$

On adding Eqs. (i), (ii) and (iii), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2014} \quad \text{…(iv)}$$

From Eq. (i), $2xz - 2yz + 1 = \frac{z}{2014}$ …(v)

From Eq. (ii), $2xy - 2zx + 1 = \frac{x}{2014}$ …(vi)

From Eq. (iii), $2zy - 2xy + 1 = \frac{y}{2014}$ …(vii)

On adding Eqs. (v), (vi) and (vii), we get $3 = \frac{x + y + z}{2014}$

$$\Rightarrow \quad x + y + z = 6042 \quad \text{…(viii)}$$

On multiplying Eqs. (iv) and (vii), we get

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 9$$

It is possible only, when $x = y = z$

We know that for x, y, z,

$$\text{AM} \geq \text{HM}$$

$$\Rightarrow \quad \frac{x + y + z}{3} \geq \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$$

$$\Rightarrow \quad (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$$

$$\Rightarrow \quad (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 9$$

If only when terms are equal.

On putting, $x = y = z = a$ in Eq. (i), we get

$$2a - 2a + \frac{1}{a} = \frac{1}{2014} \Rightarrow \quad a = 2014$$

$$\therefore \quad x = y = z = 2014$$

2. We have,

$$(x+y)^{2/3} + 2(x-y)^{2/3} = 3(x^2-y^2)^{1/3} \quad \ldots(i)$$

and $$3x - 2y = 13 \quad \ldots(ii)$$

Divide each term of Eq. (i) by $(x^2-y^2)^{1/3}$, we get

$$\left(\frac{x+y}{x-y}\right)^{1/3} + 2\left(\frac{x-y}{x+y}\right)^{1/3} = 3$$

This equation is a quadratic in $\left(\frac{x+y}{x-y}\right)^{1/3}$ from which we easily find $\left(\frac{x+y}{x-y}\right)^{1/3} = 2$ or 1.

$$\Rightarrow \quad \frac{x+y}{x-y} = 8 \text{ or } 1 \Rightarrow 7x = 9y \text{ or } y = 0$$

Combining these equations with Eq. (ii), we obtain

$$x = 9, y = 7 \text{ or } x = \frac{13}{3}, y = 0$$

3. Since, $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$$\Rightarrow \quad ab+bc+ca = \frac{(1^2-9)}{2} = -4$$

Also,

$$a^3+b^3+c^3-3abc$$
$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$
$$= 1\{9-(-4)\}$$
$$= 13$$

So that $abc = \frac{(1-13)}{3} = -4$

Now, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{ab+bc+ca}{abc} = \frac{(-4)}{(-4)} = 1$

4. Since, the students is allowed to select atmost n books out of $(2n+1)$ books so that number of selections

$$= {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots + {}^{2n-1}C_n = 63 \quad \ldots(i)$$

Now, $${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

$$\Rightarrow \quad 1 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) + ({}^{2n+1}C_{2n+1-(n+1)} + {}^{2n+1}C_{2n+1-(n+2)} + {}^{2n+1}C_{2n+1-(n+3)} + \ldots + {}^{2n+1}C_{2n+1-2n}) = 2^{2n+1}$$

$$\Rightarrow \quad 2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) = 2^{2n+1}$$

$$\Rightarrow \quad 2 + 2 \cdot 63 = 2^{2n+1}$$

$$\Rightarrow \quad 2 + 126 = 2^{2n+1}$$

$$\Rightarrow \quad 2^{2n+1} = 128$$

$$\Rightarrow \quad 2^{2n} = 64 = 2^6 \text{ or } 2n = 6$$

$$\Rightarrow \quad n = 3$$

5. Let the equations of three tangents is

$$y = m_1x + \frac{a}{m_1} \quad \ldots(i)$$

$$y = m_2x + \frac{a}{m_2} \quad \ldots(ii)$$

and $$y = m_3x + \frac{a}{m_3} \quad \ldots(iii)$$

The point of intersection of Eqs. (ii) and (iii) is found, by solving them, to be $\left[\frac{a}{m_2m_3}, a\left(\frac{1}{m_2}+\frac{1}{m_3}\right)\right]$.

The equation to the straight line through this point perpendicular to Eq. (i) is

$$y - a\left(\frac{1}{m_2}+\frac{1}{m_3}\right) = -\frac{1}{m_1}\left(x - \frac{a}{m_2m_3}\right)$$

i.e. $$y + \frac{x}{m_1} = a\left[\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_1m_2m_3}\right] \quad \ldots(iv)$$

Similarly, the equation of the straight line through the intersection of Eqs. (iii) and (i) perpendicular to Eq. (ii) is

$$y + \frac{x}{m_2} = a\left(\frac{1}{m_3}+\frac{1}{m_1}+\frac{1}{m_1m_2m_3}\right) \quad \ldots(v)$$

and the equation to the straight line through the intersection of Eqs. (i) and (ii) perpendicular to Eq. (iii) is

$$y + \frac{x}{m_3} = a\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_1m_2m_3}\right) \quad \ldots(vi)$$

The point which is common to the straight lines (iv), (v) and (vi), i.e. the orthocentre of the triangle, is easily seen to be the point whose coordinates are

$$x = -a, y = a\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_1m_2m_3}\right)$$

and this point lies on the directrix.

6. Transforming to polar coordinates, the equation gives

$$m(\cos^3\theta - 3\cos\theta\sin^2\theta) + \sin^3\theta - 3\cos^2\theta\sin\theta = 0$$

$$\Rightarrow \quad m(1-3\tan^2\theta) + \tan^3\theta - 3\tan\theta = 0$$

$$\Rightarrow \quad m = \frac{3\tan\theta - \tan^3\theta}{1-3\tan\theta} = \tan 3\theta$$

If $m = \tan\alpha$, this equation gives $\tan 3\theta = \tan\alpha$

$$\Rightarrow \quad 3\theta = \alpha \text{ or } 180^\circ + \alpha \text{ or } 360^\circ + \alpha$$

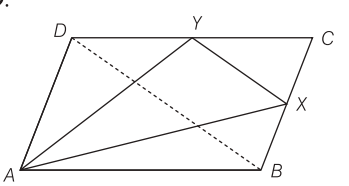
$$\Rightarrow \quad \theta = \frac{\alpha}{3} \text{ or } 60^\circ + \frac{\alpha}{3} \text{ or } 120^\circ + \frac{\alpha}{3}$$

The locus is therefore, three straight lines through the origin inclined at angles $\frac{\alpha}{3}$, $60^\circ + \frac{\alpha}{3}$ and $120^\circ + \frac{\alpha}{3}$ to the axis of X.

Hence, they are equally inclined to each other.

7. Firstly, join BD.

$\because X$ and Y are mid-points of side BC and CD respectively in $ABCD$.



$\therefore \quad XY \parallel BD$

and $\quad XY = \frac{1}{2}BD$

$\Rightarrow \quad \text{ar}(\Delta CYX) = \frac{1}{4}\text{ar}(\Delta DBC)$

$= \frac{1}{8}\text{ar}(\parallel\text{gm } ABCD)$ ...(i)

$\left[\because \frac{1}{2}\text{ar}(\parallel\text{gm } ABCD) = \text{ar}(\Delta BCD)\right]$

$\because \parallel$gm $ABCD$ and ΔABX are between same parallel lines AD and BD and $BX = \frac{1}{2}BC$.

$\therefore \text{ar}(\Delta ABX) = \frac{1}{4}\text{ar}(\parallel\text{gm } ABCD)$ ...(ii)

Similarly, $\text{ar}(\Delta AYO) = \frac{1}{4}\text{ar}(\parallel\text{gm } ABCD)$ ...(iii)

Now, $\text{ar}(\Delta AXY) = \text{ar}(\parallel\text{gm } ABCD) - [\text{ar}(\Delta ABX) + \text{ar}(\Delta AYD) + \text{ar}(\Delta CYX)]$

$= \text{ar}(\parallel\text{gm } ABCD) - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{8}\right)\text{ar}(\parallel\text{gm } ABCD)$

$= \text{ar}(\parallel\text{gm } ABCD) - \frac{5}{8}\text{ar}(\parallel\text{gm } ABCD)$

$= \frac{3}{8}\text{ar}(\parallel\text{gm } ABCD)$ **Hence proved.**

8. We have,

$f(x + y) = f(x) \cdot f(y)$ ...(i)

Put $x = y = 0$, then $f(0) = [f(0)]^2$

$\Rightarrow \quad f(0)\{f(0) - 1\} = 0$

$\Rightarrow \quad f(0) = 1 \quad [\because f(0) \neq 0]$

Putting $y = -x$ in Eq. (i), we get

$f(0) = f(x) \cdot f(-x)$

$\Rightarrow \quad 1 = f(x) \cdot f(-x)$

$\therefore \quad f(-x) = \frac{1}{f(x)}$

Also, $\quad F(x) = \frac{f(x)}{1 + [f(x)]^2}$

$\therefore \quad F(-x) = \frac{f(-x)}{1 + [f(-x)]^2} = \frac{\frac{1}{f(x)}}{1 + \frac{1}{[f(x)]^2}}$

$= \frac{f(x)}{1 + [f(x)]^2} = F(x)$

Hence, $F(x)$ is an even function.

9. $\because \quad (\sqrt{12} - \sqrt{11})(\sqrt{12} + \sqrt{11}) = 1$

and $\quad (\sqrt{7} - \sqrt{6})(\sqrt{7} + \sqrt{6}) = 1$

$\therefore \quad (\sqrt{12} - \sqrt{11})(\sqrt{12} + \sqrt{11}) = (\sqrt{7} - \sqrt{6})(\sqrt{7} + \sqrt{6})$

i.e. $\quad \frac{\sqrt{12} - \sqrt{11}}{\sqrt{7} - \sqrt{6}} = \frac{\sqrt{7} + \sqrt{6}}{\sqrt{12} + \sqrt{11}}$

$\because \quad \sqrt{7} + \sqrt{6} < 6, \sqrt{12} + \sqrt{11} > 6$

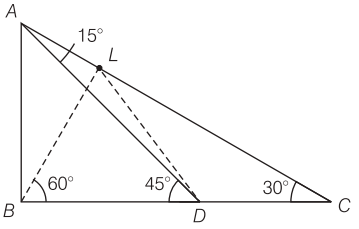
$\therefore \quad \frac{\sqrt{7} + \sqrt{6}}{\sqrt{12} + \sqrt{11}} < 6 \cdot \frac{1}{6} = 1$

Consequently, $\frac{\sqrt{12} - \sqrt{11}}{\sqrt{7} - \sqrt{6}} < 1$

So that, $(\sqrt{7} - \sqrt{6}) > (\sqrt{12} - \sqrt{11})$

10. Draw BL perpendicular to AC and join L to D. Since $\angle BCL = 30°$, we get $\angle CBL = 60°$. Since BLC is a right triangle with $\angle BCL = 30°$, we have $BL = \frac{BC}{2} = BD$.

Thus, in ΔBLD, we observe that $BL = BD$ and $\angle DBL = 60°$. This implies that BLD is an equilateral triangle and hence $LB = LD$. Using $\angle LDB = 60°$ and $\angle ADB = 45°$, we get $\angle ADL = 15°$. But $\angle DAL = 15°$. Thus, $LD = LA$. We hence have $LD = LA = LB$. This implies that L is the circumcentre of the triangle.



Thus, $\angle BAD = \frac{1}{2}\angle BLD = \frac{1}{2} \times 60° = 30°$

Continued from Page 70

24. If $f(x) = mx + c$ and $f(0) = f'(0) = 1$, what is the value of $f(2)$?

25. Differentiate the function $f(x) = \sqrt{2x + 3}$ with respect to x from first principles.

26. Differentiate the given function with respect to x, $f(x) = \frac{x + \sin x}{x + \cos x}$.

Long Answer Type Questions [6 Marks]

27. Prove that the lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent. [HOTS]

28. Evaluate $\lim_{x \to 0} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$. [HOTS]

29. Prove that $\lim_{x \to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} = -4$. [NCERT Exemplar]

30. Differentiate $e^{\sqrt{\tan x}}$ with respect to x from first principles. [HOTS]

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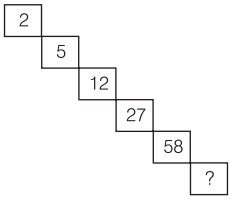
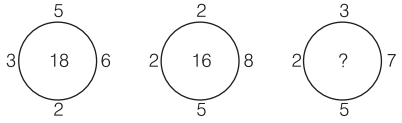
QUIZZER (NO. 24)

1. ( ) + ( ) + ( ) + ( ) + ( ) = 30
   Fill the brackets using the numbers 1, 3, 5, 7, 9, 11, 13 and 15.
2. Can you find the last number in the number sequence given below?
   10 : 10 : 20 : 45 : 110 : 300 : ?
3. A 51% tail bias coin is used to toss a 52% bias for head in order to get fair result. In such situation, can you calculate the bias of the floor?
4. If one and a half women, drink one and a half tea in one and a half minutes. How many tea can 9 women drink in 3 min?
5. There is an integer, which holds a very special property. If you subtract any number from that integer, the resultant is exactly divisible by the successor of that number.
6. A, P, R, X, S and Z are sitting in a row. S and Z are in the centre and A and P are at the ends. R is sitting on the left of A. Then, who is sitting on the right of P?
7. What number replaces the question mark?



8. A clock is set at 5 am. The clock losses 16 min in 24 h. What will be the true time when the clock indicates 10 pm on 4th day?
9. Find the missing number in the given responses.



10. In a certain code CRAYON is coded in the order $C = 1$; $R = 2$; $A = 3$ and so on. How will you express 2nd May, 1634?

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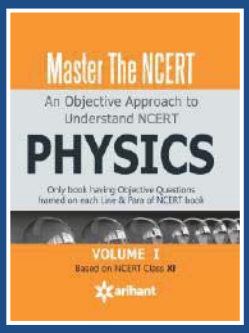

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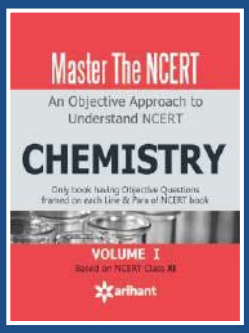

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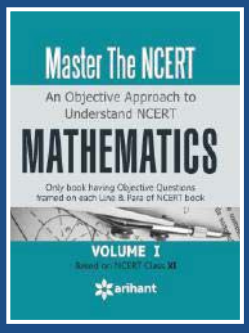

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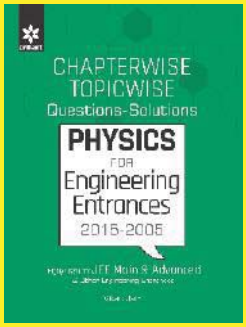

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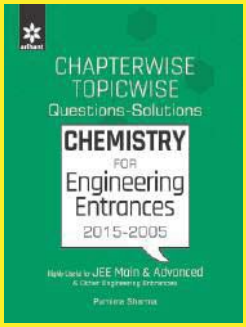

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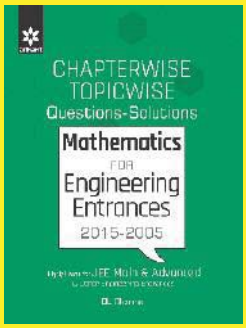

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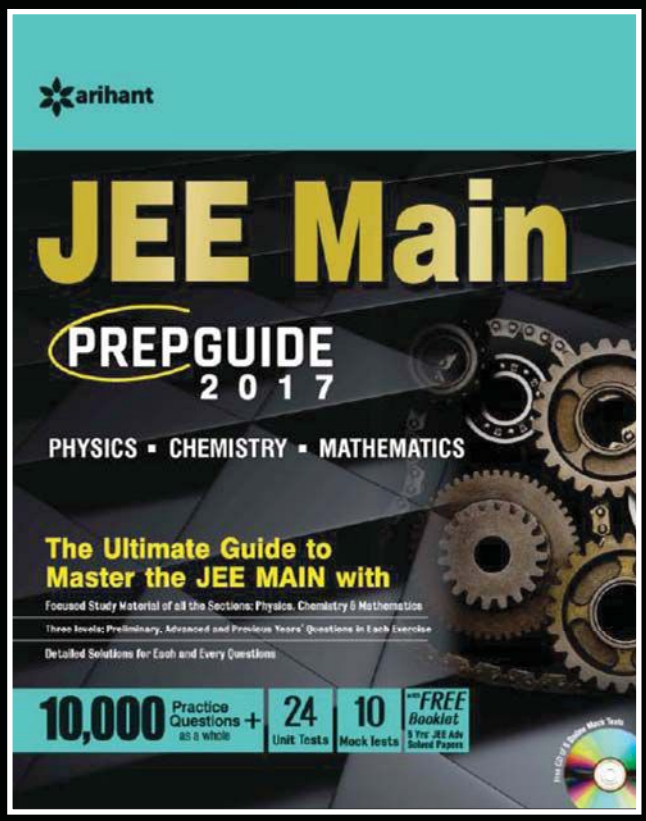
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